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Analytic Expressions for Singular Vectors of the $N = 2$ Superconformal Algebra

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ABSTRACT

Using explicit expressions for a class of singular vectors of the $N = 2$ (untwisted) algebra and following the approach of Malikov-Feigin-Fuchs and Kent, we show that the analytically extended Verma modules contain two linearly independent neutral singular vectors at the same grade. We construct this two dimensional space and we identify the singular vectors of the original Verma modules. We show that in some Verma modules these expressions lead to two linearly independent singular vectors which are at the same grade and have the same charge.

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1. Introduction

The highest weight representations of the Virasoro algebra play a crucial rôle in analysing conformal field theories. In most cases these representations contain singular vectors which lead to differential equations for the correlation functions and hence describe the dynamics of the system. Benoit and Saint-Aubin[3] gave explicit expressions for a class of the Virasoro singular vectors (the *BSA Virasoro singular vectors*). Using these results, Bauer, Di Francesco, Itzykson and Zuber developed a recursive method to compute all the Virasoro singular vectors[1, 2], the so called *fusion method*. This method can be used to give explicit formulae for the Virasoro singular vectors[16]. A completely different approach to this problem is the *analytic continuation method* which was developed by Malikov, Feigin and Fuchs for Kac-Moody algebras[13] and was extended to the Virasoro algebra by Kent[11]. Recently, Ganchev and Petkova developed a third method which transforms Kac-Moody singular vectors into Virasoro ones[10, 15].

In a recent paper[6] we used the fusion method of Bauer et al. to find the analogues of the BSA Virasoro singular vectors for the $N = 2$ (untwisted) algebra. In theory the same method can be applied to obtain all uncharged singular vectors, but this turns out to be even more complicated than in the Virasoro case. It is however possible and of independent interest to use the analytic continuation method to find product formulae for all singular vectors, as we show in this paper.

The paper is organised in the following way: after a brief review of the $N = 2$ BSA analogue singular vectors in Sec. 2 we analytically continue the $N = 2$ algebra in Sec. 3. Sec. 4 extends the notion of singular vectors to this generalised $N = 2$ algebra which will lead in Sec. 5 to product expressions for all singular vectors of the $N = 2$ (untwisted) algebra. In Sec. 6 we show that these product expressions follow similar relations as in the Virasoro case. We then find in Sec. 7 and Sec. 8 that there can be two linearly independent singular vectors at the same grade having the same $U(1)$-charge.

2. Definitions and conventions

Let[a] $\mathtt{sc}(2)$ be the $N = 2$ (untwisted) superconformal algebra in the Neveu-Schwarz (or antiperiodic) moding, which is given by the Virasoro algebra, the Heisenberg algebra plus two anticommuting subalgebras with the (anti-)commutation relations:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{C}{12}\,(m^3 - m)\,\delta_{m+n,0} \;, \\
[L_m, G_r^{\pm}] &= (\frac{1}{2}m - r)G_{m+r}^{\pm} \;, \\
[L_m, T_n] &= -nT_{m+n} \;, \\
[T_m, T_n] &= \frac{1}{3}Cm\delta_{m+n,0} \;, \\
[T_m, G_r^{\pm}] &= \pm G_{m+r}^{\pm} \;,
\end{aligned} \tag{2.1}$$

[a] There has not been any standard notation in the literature for superconformal algebras.

$$
\begin{aligned}
\{G^+_r, G^-_s\} &= 2L_{r+s} + (r-s)T_{r+s} + \frac{C}{3}(r^2 - \frac{1}{4})\delta_{r+s,0} \ , \\
[L_m, C] &= [T_m, C] \ = \ [G^\pm_r, C] \ = \ 0 \ , \\
\{G^+_r, G^+_s\} &= \{G^-_r, G^-_s\} = 0, \qquad m,n \in \mathbb{Z}, \quad r,s \in \mathbb{Z}_{\frac{1}{2}} \ .
\end{aligned}
$$

We can write $\mathtt{sc}(2)$ in its triangular decomposition: $\mathtt{sc}(2) = \mathtt{sc}(2)_- \oplus \mathcal{H}_2 \oplus \mathtt{sc}(2)_+$, where $\mathcal{H}_2 = \mathtt{span}\{L_0, T_0, C\}$ is the Cartan subalgebra, and[b]

$$\mathtt{sc}(2)_\pm = \mathtt{span}\{L_{\pm n}, T_{\pm n}, G^+_{\pm r}, G^-_{\pm r} : n \in \mathbb{N}, r \in \mathbb{N}_{\frac{1}{2}}\}.$$

A simultaneous eigenvector $|h,q,c\rangle$ of $\mathcal{H}_2$ with L_0, T_0 and C eigenvalues h, q and c respectively and vanishing $\mathtt{sc}(2)_+$ action $\mathtt{sc}(2)_+ |h,q,c\rangle = 0$, is called a highest weight vector. The Verma module $\mathcal{V}_{h,q,c}$ is defined as the $\mathtt{sc}(2)$ left module $U(\mathtt{sc}(2)) \otimes_{\mathcal{H}_2 \oplus \mathtt{sc}(2)_+} |h,q,c\rangle$, where $U(\mathtt{sc}(2))$ denotes the universal enveloping algebra of $\mathtt{sc}(2)$. This means $\mathcal{V}_{h,q,c}$ is the representation of $\mathtt{sc}(2)$ with the basis

$$
\begin{aligned}
&\mathcal{B}_{h,q,c} = \Big\{ L_{-i_I} \ldots L_{-i_1} T_{-k_K} \ldots T_{-k_1} G^+_{-j^+_{J^+}} \ldots G^+_{-j^+_1} G^-_{-j^-_{J^-}} \ldots G^-_{-j^-_1} |h,q,c\rangle \ : \\
&i_I \geq \ldots \geq i_1 \geq 1, \ j^+_{J^+} > \ldots > j^+_1 \geq \frac{1}{2}, \ j^-_{J^-} > \ldots > j^-_1 \geq \frac{1}{2}, \ k_K \geq \ldots \geq k_1 \geq 1 \Big\}.
\end{aligned}
$$

Finally, we call a vector singular in $\mathcal{V}_{h,q,c}$, if it is not proportional to the highest weight vector but still satisfies the highest weight vector conditions[c]: $\Psi_{n,p} \in \mathcal{V}_{h,q,c}$ is called singular if $L_0 \Psi_{n,p} = (h+n)\Psi_{n,p}$, $T_0 \Psi_{n,p} = (q+p)\Psi_{n,p}$ and $\mathtt{sc}(2)_+ \Psi_{n,p} = 0$ for some $n \in \mathbb{N}$ and $p \in \mathbb{Z}$. If a vector is an eigenvector of L_0 we call its eigenvalue h its *conformal weight* and similarly its eigenvalue of T_0 is called its $U(1)$-*charge*[d].

The determinant formula given by Boucher, Friedan and Kent[4] makes it apparent that the Verma module $\mathcal{V}_{h_{r,s}(t,q),q,c(t)}$ has for positive, integral r and positive, even s an uncharged singular vector at grade[e] $\frac{rs}{2}$ which we want to call $\Psi_{r,s}$. We use the parametrisation:

$$
\begin{aligned}
c(t) &= 3 - 3t \ , \\
h_{r,s}(t,q) &= \frac{(s-rt)^2}{8t} - \frac{q^2}{2t} - \frac{t}{8} \ .
\end{aligned} \tag{2.2}
$$

We can find ± 1 charged singular vectors $\Psi^\pm_k$ in the Verma module $\mathcal{V}_{h^\pm_k(t,q),q,c(t)}$ at grade k for $k \in \mathbb{N}_{\frac{1}{2}}$. The conformal weight $h^\pm_k$ is:

$$h^\pm_k(t,q) = \pm kq + \frac{1}{2}t(k^2 - \frac{1}{4}) \ . \tag{2.3}$$

In an earlier paper [6] we gave explicit expressions for $\Psi^\pm_k$ and for $\Psi_{r,2}$ by using the fusion method. In each case we can freely choose the fusion point. For instance in the

[b] We write $\mathbb{N}$ for $\{1,2,3,\ldots\}$, $\mathbb{N}_0$ for $\{0,1,2,\ldots\}$, $\mathbb{N}_{\frac{1}{2}}$ for $\{\frac{1}{2},\frac{3}{2},\frac{5}{2},\ldots\}$ and $\mathbb{Z}_{\frac{1}{2}}$ for $\{\ldots,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\ldots\}$.

[c] $\Psi \in \mathcal{V}_{h,q,c}$ automatically satisfies $C\Psi = c\Psi$.

[d] For a singular vector $\Psi_{n,p} \in \mathcal{V}_{h,q,c}$ we may simply say its charge p and its grade n rather than $U(1)$-charge $q+p$ and conformal weight $h+n$.

[e] Among physicists the term "level" is also used instead of grade.

case of $\Psi_{r,2}$ we considered the three-point function $\langle 0|\,\Phi_{h_{r,2}(t,q),q,c(t)}(Z_f)\;\Phi_{h_{r,0}(t,q),q,c(t)}(Z_1)$ $\Phi_{h_{1,2}(t,0),0,c(t)}(Z_2)\,|0\rangle$ where we have the freedom of choosing the relative position of the points Z_1, Z_2 and Z_f. Therefore we introduced in ref. [6] the fusion point parameter η: $Z_f = Z_2 + \eta(Z_1 - Z_2)$. For $\eta = 1$ we can write these singular vectors in the following form:

$$\Psi_{r,2} = (1,0,0,0) \sum_{\substack{j=2 \\ j \text{ even}}}^{2r} \sum_{\substack{n_1+\ldots+n_j=r \\ n_i\in\mathbb{N}_{\frac{1}{2}}}} E'_{n_j+\frac{1}{2}}(r) T_{n_{j-1}+\frac{1}{2}}(r-n_j) E_{n_{j-2}+\frac{1}{2}}(n_1+\ldots+n_{j-2})\ldots$$
$$\ldots T_{n_3+\frac{1}{2}}(n_1+n_2+n_3) E_{n_2+\frac{1}{2}}(n_1+n_2) T_{n_1+\frac{1}{2}}(n_1) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} |h_{r,2}(t,q),q,c(t)\rangle \;, \quad (2.4)$$

where $E_k(n)$, $E'_k(n)$ and $T_k(s)$ are four-by-four matrices ($k \in \mathbb{N}$):

$$E'_k(n) = (-1)^k \begin{pmatrix} 2(L_{-k}+\frac{q}{t}T_{-k}) & -T_{-k} & -G^+_{-k+\frac{1}{2}} & -G^-_{-k+\frac{1}{2}} \\ -n(n-r)T_{-k} & 0 & 0 & 0 \\ 0 & 0 & -nt(n-r)\delta_{k,1} & 0 \\ 0 & 0 & 0 & -nt(n-r)\delta_{k,1} \end{pmatrix} \;, \quad (2.5)$$

$$E_k(n) = \frac{1}{nt(n-r)} E'_k(n) \;, \quad (2.6)$$

$$T_k(s) = (-1)^k \begin{pmatrix} -\delta_{k,1} & 0 & 0 & 0 \\ 0 & -\delta_{k,1} & 0 & 0 \\ \frac{G^-_{-k+\frac{1}{2}}}{q-1+(\frac{r}{2}-s)t} & 0 & \frac{-T_{-k}}{q-1+(\frac{r}{2}-s)t} & 0 \\ \frac{-G^+_{-k+\frac{1}{2}}}{q+1-(\frac{r}{2}-s)t} & 0 & 0 & \frac{-T_{-k}}{q+1-(\frac{r}{2}-s)t} \end{pmatrix} \;. \quad (2.7)$$

Using the same parameter η, the odd singular vectors $\Psi^{\pm}_k$ can be written as follows:

$$\Psi^+_k = (0,0,0,1) \sum_{\substack{j=1 \\ j \text{ odd}}}^{2k} \sum_{\substack{n_1+\ldots+n_j=k \\ n_i\in\mathbb{N}_{\frac{1}{2}}}} T^{+\prime}_{n_j+\frac{1}{2}}(k) E^+_{n_{j-1}+\frac{1}{2}}(r-n_j) T^+_{n_{j-2}+\frac{1}{2}}(n_1+\ldots+n_{j-2})\ldots$$
$$\ldots T^+_{n_3+\frac{1}{2}}(n_1+n_2+n_3) E^+_{n_2+\frac{1}{2}}(n_1+n_2) T^+_{n_1+\frac{1}{2}}(n_1) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \left| h^+_k(t,q),q,c(t) \right\rangle \;, \quad (2.8)$$

$$\Psi^-_k = (0,0,1,0) \sum_{\substack{j=1 \\ j \text{ odd}}}^{2k} \sum_{\substack{n_1+\ldots+n_j=k \\ n_i\in\mathbb{N}_{\frac{1}{2}}}} T^{-\prime}_{n_j+\frac{1}{2}}(k) E^-_{n_{j-1}+\frac{1}{2}}(r-n_j) T^-_{n_{j-2}+\frac{1}{2}}(n_1+\ldots+n_{j-2})\ldots$$
$$\ldots T^-_{n_3+\frac{1}{2}}(n_1+n_2+n_3) E^-_{n_2+\frac{1}{2}}(n_1+n_2) T^-_{n_1+\frac{1}{2}}(n_1) \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \left| h^-_k(t,q),q,c(t) \right\rangle \;, \quad (2.9)$$

where the four-by-four matrices $E_j^{\pm}(n)$, $T_j^{\pm}(s)$ and $T_j^{\pm\prime}(s)$ are ($j \in \mathbb{N}$):

$$E_j^{\pm}(n) = (-1)^j$$

$$\begin{pmatrix} -\frac{2L_{-j}+[(2k-n)(2q\pm1)\pm4\frac{q}{t}(q\pm1)]T_{-j}}{n[(2k-n)t\pm2(q\pm1)]} & \frac{T_{-j}}{n[(2k-n)t\pm2(q\pm1)]} & \frac{G^{+}_{-j+1/2}}{n[(2k-n)t\pm2(q\pm1)]} & \frac{G^{-}_{-j+1/2}}{n[(2k-n)t\pm2(q\pm1)]} \\ -\frac{2(2q\pm1)L_{-j}+[(2k-n)-4\frac{q}{t}(q\pm1)]T_{-j}}{(2k-n)t\pm2(q\pm1)} & \frac{(2q\pm1)T_{-j}}{(2k-n)t\pm2(q\pm1)} & \frac{(2q\pm1)G^{+}_{-j+1/2}}{(2k-n)t\pm2(q\pm1)} & \frac{(2q\pm1)G^{-}_{-j+1/2}}{(2k-n)t\pm2(q\pm1)} \\ 0 & 0 & -\delta_{j,1} & 0 \\ 0 & 0 & 0 & -\delta_{j,1} \end{pmatrix}, \tag{2.10}$$

$$T_j^{+}(s) = (-1)^j \begin{pmatrix} -\delta_{j,1} & 0 & 0 & 0 \\ 0 & -\delta_{j,1} & 0 & 0 \\ -\frac{2qG^{-}_{-j+1/2}}{(k-s)t+2q} & 0 & \frac{2qT_{-j}}{(k-s)t+2q} & 0 \\ \frac{2(q+1)G^{+}_{-j+1/2}}{(k-s)t} & 0 & 0 & \frac{2(q+1)T_{-j}}{(k-s)t} \end{pmatrix}, \tag{2.11}$$

$$T_j^{-}(s) = (-1)^j \begin{pmatrix} -\delta_{j,1} & 0 & 0 & 0 \\ 0 & -\delta_{j,1} & 0 & 0 \\ -\frac{2(q-1)G^{-}_{-j+1/2}}{(k-s)t} & 0 & \frac{2(q-1)T_{-j}}{(k-s)t} & 0 \\ \frac{2qG^{+}_{-j+1/2}}{(k-s)t-2q} & 0 & 0 & \frac{2qT_{-j}}{(k-s)t-2q} \end{pmatrix}, \tag{2.12}$$

$$T_j^{\pm\prime}(s) = (k-s)tT_j^{\pm}(s) \quad . \tag{2.13}$$

In the following sections we will use these vectors to obtain product formulae for the singular vectors $\Psi_{r,s}$ in terms of analytically continued expressions for $\Psi_{r,2}$ and $\Psi_k^{\pm}$.

3. The analytically extended $\mathtt{sc}(2)$ algebra

In the manner of Malikov, Feigin, Fuchs[13] and Kent[4] we extend the algebra $\mathtt{sc}(2)$ to include operators of the form[f] L_{-1}^{a} for $a \in \mathbb{C}$. This extension corresponds to an underlying pseudodifferential structure[4] for the even sector but not for the odd sector. We define $\widetilde{\mathtt{sc}(2)}$ to be the vector space[g] which contains the generators $\{L_n, G_r^{+}, G_r^{-}, T_n, L_{-1}^{a}, C; n \in \mathbb{Z}, r \in \mathbb{Z}_{\frac{1}{2}}, a \in \mathbb{C}\}$

[f] Instead of introducing operators of the form L_{-1}^{a} we could have equally well chosen T_{-1}^{a}. However L_{-1}^{a} turns out to be more appropriate as we shall see later on.

[g] The supercommutator of two elements of $\widetilde{\mathtt{sc}(2)}$ can not always be written as a linear combination of generators, therefore $\widetilde{\mathtt{sc}(2)}$ does not define a superalgebra. The term "non-linear" algebra is sometimes used by physicists.

and on which the supercommutator is definined and satisfies the (anti-)commutation relations (2.1) and in addition[h]:

$$
\begin{aligned}
[L_m, L_{-1}^a] &= \sum_{i=1}^{\infty} \begin{pmatrix} a \\ i \end{pmatrix} (m+1)^{\underline{i}} L_{-1}^{a-i} L_{m-i}, && m \geq 0, m \in \mathbb{Z} \ , \\
[L_{-1}^a, L_m] &= \sum_{i=1}^{\infty} (-1)^i \begin{pmatrix} a \\ i \end{pmatrix} (m+1)^{\underline{i}} L_{m-i} L_{-1}^{a-i}, && m < 0, m \in \mathbb{Z} \ , \\
[T_m, L_{-1}^a] &= \sum_{i=1}^{\infty} \begin{pmatrix} a \\ i \end{pmatrix} (m)^{\underline{i}} L_{-1}^{a-i} T_{m-i}, && m \geq 0, m \in \mathbb{Z} \ , \\
[L_{-1}^a, T_m] &= \sum_{i=1}^{\infty} (-1)^i \begin{pmatrix} a \\ i \end{pmatrix} (m)^{\underline{i}} T_{m-i} L_{-1}^{a-i}, && m < 0, m \in \mathbb{Z} \ , \\
[G_r^{\pm}, L_{-1}^a] &= \sum_{i=1}^{\infty} \begin{pmatrix} a \\ i \end{pmatrix} (r+\tfrac{1}{2})^{\underline{i}} L_{-1}^{a-i} G_{r-i}^{\pm}, && r > 0, r \in \mathbb{Z}_{\frac{1}{2}} \ , \\
[L_{-1}^a, G_r^{\pm}] &= \sum_{i=1}^{\infty} (-1)^i \begin{pmatrix} a \\ i \end{pmatrix} (r+\tfrac{1}{2})^{\underline{i}} G_{r-i}^{\pm} L_{-1}^{a-i}, && r < 0, r \in \mathbb{Z}_{\frac{1}{2}} \ , \\
[L_{-1}^a, C] &= 0 \ , \\
L_{-1}^1 = L_{-1} \ , \quad L_{-1}^a L_{-1}^b &= L_{-1}^{a+b}, && a, b \in \mathbb{C} \ .
\end{aligned}
\tag{3.1}
$$

We point out that these commutation relations are not completely arbitrarily chosen. For integral a they have to coincide with EQS. (2.1) and for $a \in \mathbb{C}$ we use[i]

$$
\begin{aligned}
[A^a, B] &= \sum_{i=1}^{\infty} \begin{pmatrix} a \\ i \end{pmatrix} [A, B]_i \, A^{a-i} \ , \\
[A, B^a] &= \sum_{i=1}^{\infty} \begin{pmatrix} a \\ i \end{pmatrix} B^{a-i} \, {}_i[A, B] \ ,
\end{aligned}
\tag{3.2}
$$

to obtain EQS. (3.1).

It is easy to see that L_{-1}^a has the conformal weight a and the $U(1)$-charge 0 for $a \in \mathbb{C}$ with respect to the adjoint representation.

The triangular decomposition of $\widetilde{\mathtt{sc}(2)}$ is $\widetilde{\mathtt{sc}(2)} = \widetilde{\mathtt{sc}(2)}_- \oplus \mathcal{H}_2 \oplus \mathtt{sc}(2)_+$ where $\widetilde{\mathtt{sc}(2)}_-$ is $\mathtt{sc}(2)_-$ extended by the additional operators L_{-1}^a. Exactly as above we define vectors $|h, q, c\rangle$ which are simultaneously L_0, T_0 and C eigenvectors with eigenvalues h, q and c respectively and $\mathtt{sc}(2)_+ \, |h, q, c\rangle = 0$. Despite the fact that L_{-1}^{-1} lowers the weight[j] we still want to call these vectors highest weight vectors. It is straightforward to define the extended Verma module $\widetilde{\mathcal{V}}_{h,q,c}$ as $\widetilde{\mathcal{V}}_{h,q,c} = U(\widetilde{\mathtt{sc}(2)}) \otimes_{\mathcal{H}_2 \oplus \mathtt{sc}(2)_+} |h, q, c\rangle$.

[h] The falling product $(x)^{\underline{n}}$ is defined as $x(x-1)\ldots(x-n+1)$.

[i] $[A, B]_i = [A, [A, B]_{i-1}]$, $[A, B]_0 = B$ and ${}_i[A, B] = [{}_{i-1}[A, B], B]$, ${}_0[A, B] = A$.

[j] There is the usual historical confusion: what physicists call highest weight vector is actually a vector of lowest weight in the Verma module.

The vectors in $\widetilde{\mathcal{V}}_{h,q,c}$ are formal power series in L_{-1} for which we can give a basis:

$$\widetilde{\mathcal{B}}_{h,q,c} = \Big\{ L_{-i_I} \dots L_{-i_1} T_{-k_K} \dots T_{-k_1} G^+_{-j^+_{J+}} \dots G^+_{-j^+_1} G^-_{-j^-_{J-}} \dots G^-_{-j^-_1} L^a_{-1} \left| h,q,c \right\rangle \quad : \\ i_I \geq \dots \geq i_1 \geq 2,\ j^+_{J+} > \dots > j^+_1 \geq \frac{1}{2},\ j^-_{J-} > \dots > j^-_1 \geq \frac{1}{2},\ k_K \geq \dots \geq k_1 \geq 1,\ a \in \mathbb{C} \Big\}.$$

For vectors with $a = 0$ the set of basis elements $\widetilde{\mathcal{B}}^{a=0}_{h,q,c}$ decomposes in integer and half-integer L_0 grade spaces. Their operators shall be denoted by $\widetilde{\mathcal{L}}^n$:

$$\widetilde{\mathcal{L}}^n = \Big\{ L_{-i_I} \dots L_{-i_1} T_{-k_K} \dots T_{-k_1} G^+_{-j^+_{J+}} \dots G^+_{-j^+_1} G^-_{-j^-_{J-}} \dots G^-_{-j^-_1} \quad : \\ i_I \geq \dots \geq i_1 \geq 2,\ j^+_{J+} > \dots > j^+_1 \geq \frac{1}{2},\ j^-_{J-} > \dots > j^-_1 \geq \frac{1}{2},\ k_K \geq \dots \geq k_1 \geq 1, \\ i_I + \dots + i_1 + j^+_{J+} + \dots + j^+_1 + j^-_{J-} + \dots + j^-_1 + k_K + \dots + k_1 = n \Big\}.$$

We can define products of such series using the usual Cauchy product of series. However, without a norm, we cannot define the convergence of series to zero. Instead we define a slightly generalised notion of singular vectors in $\widetilde{\mathcal{V}}_{h,q,c}$.

A general element at grade a in $\widetilde{\mathcal{V}}_{h,q,c}$ is of the form:

$$\Psi_a = \lambda_0 L^a_{-1} \left| h,q,c \right\rangle + \sum_{\substack{k=\frac{1}{2} \\ k \in \mathbb{N} \cup \mathbb{N}_{\frac{1}{2}}}}^{\infty} \sum_{X_k \in \widetilde{\mathcal{L}}^k} \lambda_{X_k} X_k L^{a-k}_{-1} \left| h,q,c \right\rangle . \tag{3.3}$$

We say Ψ_b is of order $b - N$ and we write $\Psi_b = \mathcal{O}(b - N)$ if the leading term of the series contains L^{b-N}_{-1}:

$$\Psi_b = \sum_{\substack{k=N \\ k \in \mathbb{N} \cup \mathbb{N}_{\frac{1}{2}}}}^{\infty} \sum_{X_k \in \widetilde{\mathcal{L}}^k} \lambda_{X_k} X_k L^{b-k}_{-1} \left| h,q,c \right\rangle . \tag{3.4}$$

Finally, we define the sequence of cut off vectors Ψ^M_a corresponding to Ψ_a:

$$\Psi^M_a = \lambda_0 L^a_{-1} \left| h,q,c \right\rangle + \sum_{\substack{k=\frac{1}{2} \\ k \in \mathbb{N} \cup \mathbb{N}_{\frac{1}{2}}}}^{M} \sum_{X_k \in \widetilde{\mathcal{L}}^k} \lambda_{X_k} X_k L^{a-k}_{-1} \left| h,q,c \right\rangle , \quad M \in \mathbb{N} . \tag{3.5}$$

Using this notation, we say the vector $\Psi_{a,p} \in \widetilde{\mathcal{V}}_{h,q,c}$ is singular[k] if

$$\begin{aligned} L_0 \Psi_{a,p} &= (h + a) \Psi_{a,p} \quad , \ a \in \mathbb{C} \ , \\ T_0 \Psi_{a,p} &= (q + p) \Psi_{a,p} \quad , \ p \in \mathbb{Z} \ , \\ \mathtt{sc}(2)_+ \Psi^M_{a,p} &= \mathcal{O}(a - M) \quad \forall M \in \mathbb{N} \ . \end{aligned} \tag{3.6}$$

This is a generalisation of the definition we gave for singular vectors in $\mathcal{V}_{h,q,c}$, as the following two theorems show.

[k] Again, $\Psi \in \widetilde{\mathcal{V}}_{h,q,c}$ implies $C\Psi = c\Psi$.

Theorem 3.A *If $\Psi_{n,p} \in \mathcal{V}_{h,q,c}$ is singular in $\mathcal{V}_{h,q,c}$ at grade n with charge p then it is also singular in $\widetilde{\mathcal{V}}_{h,q,c}$.*

Proof: Obviously $L_0\Psi_{n,p} = (h+n)\Psi_{n,p}$ and $T_0\Psi_{n,p} = (q+p)\Psi_{n,p}$. Let us write $\Psi_{n,p}$ in the basis $\widetilde{\mathcal{B}}_{h,q,c}$:

$$\begin{aligned}
\Psi_{n,p} &= \lambda_0 L_{-1}^n |h,q,c\rangle + \sum_{\substack{k=\frac{1}{2} \\ k\in\mathbb{N}_0\cup\mathbb{N}_{\frac{1}{2}}}}^{n} \sum_{X_k\in\widetilde{\mathcal{L}}^k} \lambda_{X_k} X_k L_{-1}^{n-k} |h,q,c\rangle \quad , \\
\Psi_{n,p}^M &= \lambda_0 L_{-1}^n |h,q,c\rangle + \sum_{\substack{k=\frac{1}{2} \\ k\in\mathbb{N}_0\cup\mathbb{N}_{\frac{1}{2}}}}^{M} \sum_{X_k\in\widetilde{\mathcal{L}}^k} \lambda_{X_k} X_k L_{-1}^{n-k} |h,q,c\rangle \quad .
\end{aligned}$$

Let X be taken from $\mathtt{sc}(2)_+$. Since $\Psi_{n,p}^M - \Psi_{n,p} = \mathcal{O}(n-M-1)$ we find:

$$X\Psi_{n,p}^M = X\mathcal{O}(n-M-1) = \mathcal{O}(n-M) \quad .$$

□

Theorem 3.B *If $\Psi_{a,p} \in \widetilde{\mathcal{V}}_{h,q,c}$ is singular and finite then $\mathtt{sc}(2)_+\Psi_{a,p} \equiv 0$.*

Proof: $\Psi_{a,p}$ finite $\Rightarrow \exists m \in \mathbb{N} : \Psi_{a,p} = \Psi_{a,p}^n \quad \forall n > m \Rightarrow \forall X \in \widetilde{\mathtt{sc}(2)}_+, \quad n \in \mathbb{N}, \quad n > m :$ $X\Psi_{a,p} = X\Psi_{a,p}^n = \mathcal{O}(a-n) \Rightarrow X\Psi_{a,p} = 0.$ □

For our further discussions we need to determine the coefficients[l] $\lambda_{G^+_{-1/2}G^-_{-1/2}}$ and $\lambda_{T_{-1}}$ of $\Psi_{r,2}$. We use the notation:

$$\Lambda_2(r,2) = \lambda_{G^+_{-1/2}G^-_{-1/2}} \quad , \tag{3.7}$$

$$\Lambda_3(r,2) = \lambda_{T_{-1}} \quad . \tag{3.8}$$

Theorem 3.C *For $\Psi_{r,2}$ we normalise $\lambda_0 = 1$ then the coefficients Λ are:*

$$\Lambda_2(r,2) = \frac{1}{2}\Big[\prod_{n=1}^{r} \frac{q+1-(\frac{r}{2}-n+\frac{1}{2})t}{q-1+(\frac{r}{2}-n+\frac{1}{2})t} - 1\Big] \quad , \tag{3.9}$$

$$\Lambda_3(r,2) = \frac{q+1}{t} r \quad . \tag{3.10}$$

[l]Note that we normalised λ_0, the coefficient of L_{-1}^a, to 1.

Proof: Only the longest partition of the expression (2.4) will contribute towards $\Lambda_2(r,2)$ and $\Lambda_3(r,2)$:

$$(1,0,0,0)E_1'(r)T_1(r-\frac{1}{2})E_1(r-1)\ldots E_1(1)T_1(\frac{1}{2})(1,0,0,0)^T \ .$$

If we replace L_{-1} by $\frac{1}{2}(G^+_{-1/2}G^-_{-1/2} + G^-_{-1/2}G^+_{-1/2})$ and use $(G^{\pm}_{-1/2})^2 = 0$ then $\Lambda_2(r,2)$ is very easily verified by multiplying the components $(1,1)$ of the product $E_1(n)T_1(n-\frac{1}{2})$. Also, those matrix components are the only ones contributing towards $\Lambda_3(r,2)$. □

To identify later the correct singular vector $\Psi_{r,s}$ we need its coefficient $\Lambda_2(r,s)$. Eq. (3.9), calculations for $\Psi_{1,s}$ and computer calculations at higher grades lead to the conjecture:

$$\Lambda_2(r,s) \quad = \quad \frac{1}{2}\Big[\prod_{n=1}^{r} \frac{\frac{s-rt}{2t} + \frac{q}{t} - \frac{1}{2} + n}{-\frac{s-rt}{2t} + \frac{q}{t} + \frac{1}{2} - n} - 1\Big] \ . \tag{3.11}$$

4. Singular vectors in $\widetilde{\mathcal{V}}_{h,q,c}$

The determinant formula[4] tells us about the singular vectors in $\mathcal{V}_{h,q,c}$. In this section we investigate the generalised modules $\widetilde{\mathcal{V}}_{h,q,c}$. Using results of the following section we find that at all grades the modules $\widetilde{\mathcal{V}}_{h,q,c}$ contain two linearly independent neutral singular vectors as well as one $+1$ and one -1 charged singular vector. We can show that there are no singular vectors of charge greater than 1 or smaller than -1. In particular the module $\widetilde{\mathcal{V}}_{h_{r,s}(t,q),q,c(t)}$ contains two linearly independent neutral singular vectors at grade $\frac{rs}{2}$ and $\widetilde{\mathcal{V}}_{h^{\pm}_k(t,q),q,c(t)}$ contains one ± 1 charged singular vector at grade k.

4.1. Uniqueness of singular vectors

We take the most general uncharged vector in $\widetilde{\mathcal{V}}_{h,q,c}$ at grade $a \in \mathbb{C}$:

$$\Psi_a^{\circ} \quad = \quad \lambda_0 L_{-1}^a \,|h,q,c\rangle + \sum_{k=1}^{\infty} \sum_{\substack{X_k \in \widetilde{\mathcal{B}}^k \\ [T_0,X_k]=0}} \lambda_{X_k} X_k L_{-1}^{a-k} \,|h,q,c\rangle \ , \tag{4.1}$$

and the most general ± 1 charged vectors also at grade $a \in \mathbb{C}$:

$$\Psi_a^{\pm} \quad = \quad \sum_{\substack{k=\frac{1}{2} \\ k\in\mathbb{N}_{\frac{1}{2}}}}^{\infty} \sum_{\substack{X_k \in \widetilde{\mathcal{B}}^k \\ [T_0,X_k]=\pm X_k}} \lambda^{\pm}_{X_k} X_k L_{-1}^{a-k} \,|h,q,c\rangle \ . \tag{4.2}$$

For convenience we want to use the following notation:

Definition 4.A *Let I, J and $K^\pm$ denote the ordered sequences*

$$\begin{array}{llll} I = (i_{\|I\|}, i_{\|I\|-1}, \ldots, i_1) \ , & i_{\|I\|} \geq \ldots \geq i_1 \geq 2 \ , & i_n \in \mathbb{N} \ , & n = 1, \ldots, \| I \| \ , \\ J = (j_{\|J\|}, j_{\|J\|-1}, \ldots, j_1) \ , & j_{\|J\|} \geq \ldots \geq j_1 \geq 1 \ , & j_n \in \mathbb{N} \ , & n = 1, \ldots, \| J \| \ , \\ K^\pm = (k^\pm_{\|K^\pm\|}, \ldots, k^\pm_1) \ , & k^\pm_{\|K^\pm\|} > \ldots > k^\pm_1 > \frac{1}{2} \ , & k^\pm_n \in \mathbb{N}_{\frac{1}{2}} \ , & n = 1, \ldots, \| K^\pm \| \ , \end{array} \tag{4.3}$$

where we denote the length of these sequences by $\| I \|$, $\| J \|$ and $\| K^\pm \|$ and the sum by $|I| = \sum_{m=1}^{\|I\|} i_m$, $|J| = \sum_{m=1}^{\|J\|} j_m$ and $|K^\pm| = \sum_{m=1}^{\|K^\pm\|} k^\pm_m$. The sets of all these sequences shall be called $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}^\pm$ respectively. Furthermore we denote products of $\mathtt{sc}(2)$ operators by

$$\begin{array}{rcl} L_{-I} & = & L_{-i_{\|I\|}} L_{-i_{\|I\|-1}} \ldots L_{-i_1} \ , \\ T_{-J} & = & T_{-j_{\|J\|}} T_{-j_{\|J\|-1}} \ldots T_{-j_1} \ , \\ G^\pm_{-K^\pm} & = & G^\pm_{-k^\pm_{\|K^\pm\|}} G^\pm_{-k^\pm_{\|K^\pm\|-1}} \ldots G^\pm_{-k^\pm_1} \ . \end{array} \tag{4.4}$$

Finally, we take the sequences together:

$$\begin{array}{rcl} M = & (I, J, K^+, K^-, r^+, r^-)_a & = L_{-I} T_{-J} G^+_{-K^+} G^-_{-K^-} \left(G^+_{-\frac{1}{2}}\right)^{r^+} \left(G^-_{-\frac{1}{2}}\right)^{r^-} L^a_{-1} \ , \\ M^+ = & (I, J, K^+, K^-, r)^{+-}_a & = L_{-I} T_{-J} G^+_{-K^+} G^-_{-K^-} \left(G^-_{-\frac{1}{2}}\right)^{r} G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}} L^a_{-1} \ , \\ M^- & = (I, J, K^-, K^+, r)^{-+}_a & = L_{-I} T_{-J} G^-_{-K^-} G^+_{-K^+} \left(G^+_{-\frac{1}{2}}\right)^{r} G^-_{-\frac{1}{2}} G^+_{-\frac{1}{2}} L^a_{-1} \ , \\ & & \text{with } \ r, r^\pm \in \{0, 1\} \ . \end{array} \tag{4.5}$$

Here length, sum, grade $\mathcal{L}$ and charge $\mathcal{C}$ are defined as

$$\begin{array}{rcl} \| M \| & = & \| I \| + \| J \| + \| K^+ \| + \| K^- \| + r^+ + r^- \ , \\ \| M^\pm \| & = & \| I \| + \| J \| + \| K^+ \| + \| K^- \| + r \ , \\ |M| & = & |I| + |J| + |K^+| + |K^-| + \dfrac{r^+}{2} + \dfrac{r^-}{2} \ , \\ |M^\pm| & = & |I| + |J| + |K^+| + |K^-| + \dfrac{r}{2} \ , \\ \mathcal{L}(M) & = & |M| + a \ , \\ \mathcal{L}(M^\pm) & = & |M^\pm| + a + 1 \ , \\ \mathcal{C}(M) & = & \| K^+ \| - \| K^- \| + r^+ - r^- \ , \\ \mathcal{C}(M^\pm) & = & \| K^+ \| - \| K^- \| \mp r \ . \end{array}$$

Using these definitions, we can rewrite the basis $\widetilde{\mathcal{B}}_{h,q,c}$ as

$$\widetilde{\mathcal{B}}_{h,q,c} \quad = \quad \left\{ (I, J, K^+, K^-, r^+, r^-)_a \, |h, q, c\rangle \, ; I \in \mathcal{I}, J \in \mathcal{J}, K^\pm \in \mathcal{K}^\pm, r^\pm \in \{0, 1\}, a \in \mathbb{C} \right\} \ .$$

In fact, it seems rather unnatural to prefer the order $G^+_{-1/2} G^-_{-1/2}$ to $G^-_{-1/2} G^+_{-1/2}$. And in the following it will turn out that we should preferably choose a more symmetric basis involving $M^\pm$:

$$\begin{array}{rcl} \widetilde{\mathcal{C}}_{h,q,c} & = & \Big\{ (I, J, K^+, K^-, r)^{+-}_a \, |h, q, c\rangle \, , (I, J, K^-, K^+, r)^{-+}_a \, |h, q, c\rangle \, ; \\ & & \quad I \in \mathcal{I}, J \in \mathcal{J}, K^\pm \in \mathcal{K}^\pm, r \in \{0, 1\}, a \in \mathbb{C} \Big\} \ . \end{array}$$

The subset of $\widetilde{\mathcal{C}}_{h,q,c}$ containing the vectors at fixed grade a is denoted by $\widetilde{\mathcal{C}}^{a}_{h,q,c}$ This basis naturally decomposes into two parts:

$$\begin{aligned}
\widetilde{\mathcal{C}}^{+-}_{h,q,c} &= \Big\{(I,J,K^+,K^-,r)^{+-}_a\,|h,q,c\rangle\,; I\in\mathcal{I}, J\in\mathcal{J}, K^\pm\in\mathcal{K}^\pm, r\in\{0,1\}, a\in\mathbb{C}\Big\}\ , \\
\widetilde{\mathcal{C}}^{-+}_{h,q,c} &= \Big\{(I,J,K^-,K^+,r)^{-+}_a\,|h,q,c\rangle\,; I\in\mathcal{I}, J\in\mathcal{J}, K^\pm\in\mathcal{K}^\pm, r\in\{0,1\}, a\in\mathbb{C}\Big\}\ ,
\end{aligned}$$

and hence the Verma module decomposes as follows:

$$\begin{aligned}
\widetilde{\mathcal{V}}_{h,q,c} &= \widetilde{\mathcal{V}}^{+-}_{h,q,c}\oplus\widetilde{\mathcal{V}}^{-+}_{h,q,c}\ , \\
\widetilde{\mathcal{V}}^{+-}_{h,q,c} &= \mathtt{span}\{\widetilde{\mathcal{C}}^{+-}_{h,q,c}\}\ , \\
\widetilde{\mathcal{V}}^{-+}_{h,q,c} &= \mathtt{span}\{\widetilde{\mathcal{C}}^{-+}_{h,q,c}\}\ .
\end{aligned}$$

In this basis the vectors Ψ°_a and $\Psi^\pm_a$ can be written:

$$\begin{aligned}
\Psi^\circ_a &= \sum_{k=0}^{\infty}\ \sum_{\substack{M^+=(I,J,K^+,K^-,r)^{+-}_{a-|M^+|-1}\\ |M^+|=k\ ,\ \mathcal{C}(M^+)=0}} \lambda_{M^+}M^+\,|h,q,c\rangle \\
&\quad+\sum_{k=0}^{\infty}\ \sum_{\substack{M^-=(I,J,K^-,K^+,r)^{-+}_{a-|M^-|-1}\\ |M^-|=k\ ,\ \mathcal{C}(M^-)=0}} \lambda_{M^-}M^-\,|h,q,c\rangle\ , \qquad (4.6)\\
\Psi^\pm_a &= \sum_{k=\frac12}^{\infty}\ \sum_{\substack{M^+=(I,J,K^+,K^-,r)^{+-}_{a-|M^+|-1}\\ |M^+|=k\ ,\ \mathcal{C}(M^+)=\pm1}} \lambda^\pm_{M^+}M^+\,|h,q,c\rangle \\
&\quad+\sum_{k=\frac12}^{\infty}\ \sum_{\substack{M^-=(I,J,K^-,K^+,r)^{-+}_{a-|M^-|-1}\\ |M^-|=k\ ,\ \mathcal{C}(M^-)=\pm1}} \lambda^\pm_{M^-}M^-\,|h,q,c\rangle\ . \qquad (4.7)
\end{aligned}$$

We can easily work out the commutation relations of $G^\pm_r$, for $r\in\mathbb{N}_{\frac12}$, with $G^+_{-\frac12}G^-_{-\frac12}L^a_{-1}$ and $G^-_{-\frac12}G^+_{-\frac12}L^a_{-1}$:

$$\begin{aligned}
[G^-_r, G^+_{-\frac12}G^-_{-\frac12}L^a_{-1}]\,|h,q,c\rangle &= \Big(h+\frac{a-r+\frac12}{r+\frac12}-\frac{q}{2}+1\Big)\begin{pmatrix} a\\ r-\frac12\end{pmatrix}(r+\frac12)! \\
&\qquad G^-_{-\frac12}G^+_{-\frac12}G^-_{-\frac12}L^{a-r-\frac12}_{-1}\,|h,q,c\rangle\ , \qquad (4.8)\\
[G^+_r, G^-_{-\frac12}G^+_{-\frac12}L^a_{-1}]\,|h,q,c\rangle &= \Big(h+\frac{a-r+\frac12}{r+\frac12}+\frac{q}{2}+1\Big)\begin{pmatrix} a\\ r-\frac12\end{pmatrix}(r+\frac12)! \\
&\qquad G^+_{-\frac12}G^-_{-\frac12}G^+_{-\frac12}L^{a-r-\frac12}_{-1}\,|h,q,c\rangle\ . \qquad (4.9)
\end{aligned}$$

These commutation relations imply the following theorem:

Theorem 4.B

$$G_r^{+}\widetilde{\mathcal{V}}_{h,q,c}^{-+} \in \widetilde{\mathcal{V}}_{h,q,c}^{-+} \quad , \quad G_r^{-}\widetilde{\mathcal{V}}_{h,q,c}^{+-} \in \widetilde{\mathcal{V}}_{h,q,c}^{+-} \ , \quad r \in \mathbb{N}_{\frac{1}{2}} \ . \tag{4.10}$$

This enables us to give the main theorem of this subsection:

Theorem 4.C *Let Ψ_a° and $\Psi_a^{\pm}$ be singular in $\widetilde{\mathcal{V}}_{h,q,c}$ at grade $a \in \mathbb{C}$. For Ψ_a° we find:*

$$\lambda_{(\emptyset,\emptyset,\emptyset,\emptyset,0)_{a-1}^{+-}} = \lambda_{(\emptyset,\emptyset,\emptyset,\emptyset,0)_{a-1}^{-+}} = 0 \quad \Rightarrow \quad \Psi_a^{\circ} \equiv 0 \ . \tag{4.11}$$

And for $\Psi_a^{\pm}$:

$$\lambda_{(\emptyset,\emptyset,\emptyset,\emptyset,1)_{a-\frac{3}{2}}^{+-}} = 0 \quad \Rightarrow \quad \Psi_a^{+} \equiv 0 \ , \tag{4.12}$$

$$\lambda_{(\emptyset,\emptyset,\emptyset,\emptyset,1)_{a-\frac{3}{2}}^{-+}} = 0 \quad \Rightarrow \quad \Psi_a^{-} \equiv 0 \ . \tag{4.13}$$

Theorem 4.C tells us that at given grade a there can be at most two linearly independent neutral singular vectors, one $+1$ and one -1 charged singular vector.

In order to prepare the proof of this theorem we will introduce a partial ordering on the basis $\widetilde{\mathcal{C}}_{h,q,c}$. This is analogous to the Virasoro case[11] but turns out to be far more complicated. We first define the difference of two sequences to be the componentwise difference: $\delta(I_1, I_2) = (i_{1,\min(\|I_1\|,\|I_2\|)} - i_{2,\min(\|I_1\|,\|I_2\|)}, \ldots, i_{1,1} - i_{2,1})$. Similarly we construct the action of δ on the sequences J and $K^{\pm}$.

Definition 4.D *We say $I_1 < I_2$ if the first non-trivial element of $\delta(I_1, I_2)$, read from the right to the left, is negative. If $\delta(I_1, I_2)$ is trivial we define $I_1 < I_2$ if $\| I_1 \| > \| I_2 \|$. The same shall be defined for the sequences J and $K^{\pm}$.*

We also define the ordering indicator function on the partitions I, J and $K^{\pm}$:

Definition 4.E *The function ϵ is defined as*

$$\epsilon(I_1, I_2) = \begin{cases} +1 & I_1 > I_2 \\ 0 & I_1 = I_2 \\ -1 & I_1 < I_2 \end{cases} \ , \tag{4.14}$$

and similarly for J and $K^{\pm}$. For the numbers r we define:

$$\epsilon(r_1, r_2) = \begin{cases} +1 & r_1 < r_2 \\ 0 & r_1 = r_2 \\ -1 & r_1 > r_2 \end{cases} \ . \tag{4.15}$$

For the following two definitions we keep $a \in \mathbb{C}$ fixed.

Definition 4.F *For the basis elements M^+ we introduce a total ordering: $M_1^+ < M_2^+$ if $|M_1^+| < |M_2^+|$. In the case $|M_1^+| = |M_2^+|$ we say $M_1^+ < M_2^+$ if the first non-trivial element in the sequence $\delta(M_1^+, M_2^+) = (\epsilon(J_1, J_2), \epsilon(I_1, I_2), \epsilon(K_1^+, K_2^+), \epsilon(K_1^-, K_2^-), \epsilon(r_1, r_2))$, read from the right to the left, is negative[m]. The basis elements M^- can be ordered in exactly the same way, where we always have to exchange the rôle of $+$ and $-$.*

In other words, definition 4.F first orders M_1^+ and M_2^+ according to their sums $|M_1^+|$ and $|M_2^+|$. If $|M_1^+| = |M_2^+|$ we say $M_1^+ < M_2^+$ if $r_1 > r_2$, unless $r_1 = r_2$. Then we define $M_1^+ < M_2^+$ if $K_1^- < K_2^-$. If even this does not come to a decision due to $K_1^- = K_2^-$, we do the same with K^+: $M_1^+ < M_2^+$ if $K_1^+ < K_2^+$. For $K_1^+ = K_2^+$ we take $M_1^+ < M_2^+$ if $I_1 < I_2$. And finally, if $I_1 = I_2$: $M_1^+ < M_2^+$ if $J_1 < J_2$.

On the set of basis elements $\widehat{M}$ of the form

$$
\begin{aligned}
\widehat{M}^+ &= (I, J, K, \emptyset, r)^{+-}_{a-|M^+|-1} \ , \\
\widehat{M}^- &= (I, J, K, \emptyset, r)^{-+}_{a-|M^-|-1} \ ,
\end{aligned}
$$

we extend the ordering 4.F by:

Definition 4.G *We define $\widehat{M}_1 < \widehat{M}_2$ if $|\widehat{M}_1| < |\widehat{M}_2|$. Again, in the case that $|\widehat{M}_1| = |\widehat{M}_2|$ we say $\widehat{M}_1 < \widehat{M}_2$ if the first non-trivial element in the sequence $\delta(\widehat{M}_1, \widehat{M}_2) = (\epsilon(J_1, J_2), \epsilon(I_1, I_2), \epsilon(K_1, K_2), \epsilon(r_1, r_2))$, read from the right to the left, is negative. If this has not given a decision yet, we define $\widehat{M}_1 < \widehat{M}_2$ if $\widehat{M}_1$ is of the form $\widehat{M}^+$ and $\widehat{M}_2$ is of the form $\widehat{M}^-$.*

The ordering 4.G is consistent with 4.F, so that taking the transitive closure of the two orderings, we obtain a partial ordering on $\widetilde{\mathcal{C}}^a_{h,q,c}$ with two totally ordered chains consisting of elements of the form M^+ and M^-. We can now start the proof of theorem 4.C:

Proof: We first consider the uncharged case. The vector Ψ°_a has to have a smallest element in each of the two totally ordered chains: let M_0^+ be the smallest element with non-trivial coefficient in the chain of terms of the form M^+ and accordingly M_0^- in the chain of terms of the form M^-:

$$
\begin{aligned}
M_0^+ &= (I_0^+, J_0^+, K_0^{+,+}, K_0^{+,-}, r^+)^{+-}_{a-|M_0^+|-1} \ , \\
M_0^- &= (I_0^-, J_0^-, K_0^{-,+}, K_0^{-,-}, r^-)^{-+}_{a-|M_0^-|-1} \ .
\end{aligned}
$$

We consider first M_0^+. If $K_0^{+,-} \neq \emptyset$ we take the smallest element in $K_0^{+,-}$, $k_1^{+,-}$, and act with $G^+_{k_1^{+,-}-1}$ on the cut off vector $\Psi^{\circ m}_a$ where m is sufficiently big. We generate a term $(I_0^+, J_0^+, K_0^{+,+}, K_0^{+,-}\backslash\{k_1^{+,-}\}, r_0^+)^{+-}_{a-|M_0^+|}$. This term cannot be created by any term which is according to ordering 4.F bigger. Furthermore, due to theorem 4.B, terms of the form M^- which cannot be compared with M_0^+ using the partial ordering, cannot create such a term

[m] A crucial point for our later proof is that the operators T_n are ordered last.

either. Hence $K_0^{+,-}=\emptyset$, or otherwise we would have a contradiction to the non-triviality of the coefficient of M_0^+. In exactly the same way, we can show that for M_0^- the sequence $K_0^{+,+}$ has to be trivial. Thus, the smallest elements with non-trivial coefficients in the chains of M^+ and M^- terms have to have the form:

$$\begin{aligned} M_0^+ &= (I_0^+, J_0^+, K_0^+, \emptyset, r^+)^{+-}_{a-|M_0^+|-1} \;, \\ M_0^- &= (I_0^-, J_0^-, K_0^-, \emptyset, r^-)^{-+}_{a-|M_0^-|-1} \;. \end{aligned}$$

We now continue in the same manner using the ordering 4.G. We first assume that M_0^+ is the smaller term of M_0^+ and M_0^-. If $K_0^+\neq\emptyset$ then it contains exactly one element k^+ and $r^+=1$. We act with $G^-_{k^+-1}$ on $\Psi_a^{\circ m}$. This creates the term $(I^+,J^+,\emptyset,\emptyset,1)^{+-}_{a-|M_0^+|}$. In order to create a term of this type, terms of the form M^+ have to create another L_{-1} which is not possible for a term bigger than M_0^+. A term of the form M^- contributing towards $(I^+,J^+,\emptyset,\emptyset,1)^{+-}_{a-|M_0^+|}$, has to have $K^-=\emptyset$ and hence $r=0$. Such a term would have a sum strictly smaller than $|M_0^+|$, hence it is smaller than M_0^- which contradicts the minimality assumptions. Hence, $K^+=\emptyset$ and because Ψ_a° is neutral we find in addition[n] $r^+=0$. Let us assume now $I_0^+\neq\emptyset$. In this case we look at the smallest element i_1^+ of the sequence I_0^+. We act now with $L_{i_1^+-1}$ on $\Psi_a^{\circ m}$. Again we create a term $(I_0^+\backslash\{i_1^+\},J_0^+,\emptyset,\emptyset,0)^{+-}_{a-|M_0^+|}$ by generating an additional L_{-1}. As before we see that any other term contributing towards this term would violate the minimality of either M_0^+ or M_0^-. This implies that $I_0^+=\emptyset$. Finally, we assume $J_0^+\neq\emptyset$. Again, we take the smallest element in this sequence: j_1^+. However, since the operators T_m cannot create L_{-1} terms, we have to alter the method slightly. We act with $T_{j_1^+}$ on $\Psi_a^{\circ m}$. This creates a term, where the T_{-j_1} has been annihilated: $(\emptyset,J_0^+\backslash\{j_1^+\},\emptyset,\emptyset,0)^{+-}_{a-|M_0^+|-1}$. Since $T_{-j_1^+}$ is the only operator of the form T_m which does not commute with $T_{j_1^+}$ we find again that the only terms which could contribute would violate the minimality conditions.[o] Hence: $J_0^+=\emptyset$. If we had assumed that M_0^- was the smaller term, we could have gone through similar implications for M_0^-. This result can be summarised:

$$\begin{aligned} M_0^+ < M_0^- &\;\Rightarrow\; M_0^+ = (\emptyset,\emptyset,\emptyset,\emptyset,0)^{+-}_{a-1} \;, \\ M_0^- > M_0^+ &\;\Rightarrow\; M_0^- = (\emptyset,\emptyset,\emptyset,\emptyset,0)^{-+}_{a-1} \;. \end{aligned} \tag{4.16}$$

If we take the assumptions of theorem 4.C, then EQS. (4.16) lead to a contradiction, since M_0^+ and M_0^- are totally ordered.

A similar argument applies for $\Psi_a^\pm$, where we have to take into account that due to the charge of the vectors, $(I_0,J_0,\emptyset,\emptyset,r)^{+-}$ does not exist for Ψ_a^+ and neither does $(I_0,J_0,\emptyset,\emptyset,r)^{-+}$ for Ψ_a^-. This completes the proof of theorem 4.C. □

[n] In the charged case we obtain values for r^+ according to the charge.

[o] Note that this works as well for $j_1^+=1$ which is a strong argument for choosing analytic continuation of L_{-1} rather than T_{-1}.

4.2. Existence of singular vectors

If we act with $\mathtt{sc}(2)_+$ on the cut off vectors $\Psi_a^{\circ M}$, $\Psi_a^{\pm M}$ and require the result to be of order $a-M$ then we obtain linear homogeneous systems which we denote by $\mathcal{S}_M^\circ(a,h,q,c)$ and $\mathcal{S}_M^\pm(a,h,q,c)$ respectively. Obviously the system $\mathcal{S}_M^\circ(a,h,q,c)$ is a subsystem of $\mathcal{S}_{M+1}^\circ(a,h,q,c)$ and likewise $\mathcal{S}_M^\pm(a,h,q,c)$ is a subsystem of $\mathcal{S}_{M+1}^\pm(a,h,q,c)$. We use the parametrisation (2.2) for $\mathcal{S}_M^\circ(a,h,q,c)$ and (2.3) for $\mathcal{S}_M^\pm(a,h,q,c)$ where r,s and k are chosen to be in $\mathbb{C}$. We expect to find singular vectors at grade $a=\frac{rs}{2}$ for the neutral case and $a=k$ for the charged cases, hence we replace a accordingly. The systems $\mathcal{S}_M^\circ(r,s,q,t)$ and $\mathcal{S}_M^\pm(k,q,t)$ can be written such that all the entries are polynomials in their variables.

A matrix has exactly rank j if all subdeterminants of size $>j$ vanish and there exists at least one subdeterminant of size j which is non-trivial. Let $\mathcal{W}_M^\circ$ and $\mathcal{W}_M^\pm$ denote the set of unknowns of the systems $\mathcal{S}_M^\circ(r,s,q,t)$ and $\mathcal{S}_M^\pm(k,q,t)$ respectively. The system $\mathcal{S}_M^\circ(r,s,q,t)$ has according to theorem 3.A non-trivial solutions for all pairs (r,s) where $r,s\in\mathbb{N}$ and s is even. Hence all the subdeterminants of $\mathcal{S}_M^\circ(r,s,q,t)$ of size greater or equal the number of elements in $\mathcal{W}_M^\circ$ have to vanish for $(r,s)\in\mathbb{N}\times 2\mathbb{N}$ and since these subdeterminants are polynomials they are trivial for all $r,s\in\mathbb{C}$. We find that $\mathcal{S}_M^\circ(r,s,q,t)$ has a non-trivial solution space $\Pi_M^\circ(r,s,q,t)$. The same arguments apply for $\mathcal{S}_M^\pm(k,q,t)$; we call the non-trivial solution space $\Pi_M^\pm(k,q,t)$. Let P_Π^M be the projection operators into $\mathcal{W}_M^\circ$. We obviously have $P_\Pi^M P_\Pi^{M+n}=P_\Pi^M$ for $n\in\mathbb{N}_0$. Since $\mathcal{S}_M^\circ(r,s,q,t)\subseteq\mathcal{S}_{M+n}^\circ(r,s,q,t)$ for $n\in\mathbb{N}_0$ it is easy to see that $P_\Pi^M(\Pi_{M+n}^\circ(r,s,q,t))\in\Pi_M^\circ(r,s,q,t)$. Moreover, we can show that $P_\Pi^M(\Pi_{M+n}^\circ(r,s,q,t))$ is non-trivial: assume $P_\Pi^M(\Pi_{M+n}^\circ(r,s,q,t))=\{0\}$, then $\Psi_{\frac{rs}{2}}\in\Pi_{M+n}^\circ(r,s,q,t)$, $\Psi_{\frac{rs}{2}}\neq 0$ would have $P_\Pi^M(\Psi_{\frac{rs}{2}})=0$ and hence $\Psi_{\frac{rs}{2}}=\mathcal{O}(\frac{rs}{2}-M-1)$. The action of $\mathtt{sc}(2)_+$ on $\Psi_{\frac{rs}{2}}$ has to be of order $\frac{rs}{2}-M-n$. However the proof of theorem 4.C can be applied here in exactly the same way and we obtain $\Psi_{\frac{rs}{2}}=0$. Hence we find the following inclusion chain:

$$\Pi_M^\circ(r,s,q,t)\supseteq P_\Pi^M(\Pi_{M+1}^\circ(r,s,q,t))\supseteq P_\Pi^M(\Pi_{M+2}^\circ(r,s,q,t))\ldots\supsetneq\emptyset\ . \tag{4.17}$$

Therefore the limit of the sequence $P_\Pi^M(\Pi_{M+n}^\circ(r,s,q,t))$ for n tending to infinity exists for each $M\in\mathbb{N}_0$. We denote this non-trivial set by $\lim\Pi_M^\circ(r,s,q,t)$. Since the dimensions of the spaces $P_\Pi^M(\Pi_{M+n}^\circ(r,s,q,t))$ are integers this sequence has to be constant for sufficiently big n. We say the sequence stabilises. This allows us to prove that the projection operator P_Π^M is in fact continuous: choosing n sufficiently big in $P_\Pi^M P^{M+m}(\Pi_{M+n}^\circ(r,s,q,t))=P_\Pi^M(\Pi_{M+n}^\circ(r,s,q,t))$ we obtain $P^M(\lim\Pi_{M+m}^\circ(r,s,q,t))=\lim\Pi_M^\circ(r,s,q,t)$ for $m\in\mathbb{N}_0$. Hence the sequence $\lim\Pi_M^\circ$ defines the cut off sequence of a space of singular vectors which we shall denote by $\Pi_{r,s}^\circ$. We can obtain the same important result for $\mathcal{S}_M^\pm(k,q,t)$:

Theorem 4.H *The sequences of homogeneous linear systems $\mathcal{S}_M^\circ(r,s,q,t)$ and $\mathcal{S}_M^\pm(k,q,t)$ define non-trivial solution spaces $\Pi_M^\circ(r,s,q,t)$ and $\Pi_M^\pm(k,q,t)$ respectively. These spaces converge to non-trivial spaces of singular vectors. According to theorem* 4.C *the dimensions of the limit spaces $\Pi_{r,s}^\circ$ and $\Pi_k^\pm$ are bounded by 2 for $\Pi_{r,s}^\circ$ and by 1 for $\Pi_k^\pm$.*

In particular we have found that $\mathcal{S}_M^\pm(k,q,t)$ defines at grade k uniquely the charged singular vectors $\Psi_k^+\in\widetilde{\mathcal{V}}_{h_k^+(t,q),q,c(t)}$ and $\Psi_k^-\in\widetilde{\mathcal{V}}_{h_k^-(t,q),q,c(t)}$ for $k\in\mathbb{C}$. These vectors coincide for $k\in\mathbb{N}_{\frac{1}{2}}$

with the known singular vectors in $\mathcal{V}_{h_k^{\pm}(t,q),q,c(t)}$ and for $k \in \mathbb{C}$ they are analytic continuations of them.

Computer calculations solving $\mathcal{S}_M^{\circ}$ and $\mathcal{S}_M^{\pm}$ show that already for small M the unknowns at low grades become stable.

4.3. Theorems about singular vectors in $\widetilde{\mathcal{V}}_{h,q,c}$

An immediate consequence of theorem 4.C is:

Theorem 4.I *Assume that Ψ^1 and Ψ^2 are two neutral singular vectors both at grade a in $\widetilde{\mathcal{V}}_{h,q,c}$. If*

$$\lambda^1_{(\emptyset,\emptyset,\emptyset,\emptyset,0)^{+-}_{a-1}} \quad = \quad \lambda^2_{(\emptyset,\emptyset,\emptyset,\emptyset,0)^{+-}_{a-1}}$$

and

$$\lambda^1_{(\emptyset,\emptyset,\emptyset,\emptyset,0)^{-+}_{a-1}} \quad = \quad \lambda^2_{(\emptyset,\emptyset,\emptyset,\emptyset,0)^{-+}_{a-1}}$$

then $\Psi^1 = \Psi^2$.

Proof: We look at the vector $\Psi^{\Delta} = \Psi^1 - \Psi^2$. This vector is singular and $\lambda^{\Delta}_{(\emptyset,\emptyset,\emptyset,\emptyset,0)^{+-}_{a-1}} = \lambda^{\Delta}_{(\emptyset,\emptyset,\emptyset,\emptyset,0)^{-+}_{a-1}} = 0$. Theorem 4.C implies that $\Psi^{\Delta} = 0$. □

This enables us to identify the elements in $\Pi^{\circ}_{r,s}$ using the two coefficients $\lambda_{(\emptyset,\emptyset,\emptyset,\emptyset,0)^{+-}_{\frac{rs}{2}-1}}$ and $\lambda_{(\emptyset,\emptyset,\emptyset,\emptyset,0)^{-+}_{\frac{rs}{2}-1}}$. We shall therefore give the following definition.

Definition 4.J *A vector $\Psi \in \Pi^{\circ}_{r,s}$ is denoted by giving the two coefficients $\lambda_{(\emptyset,\emptyset,\emptyset,\emptyset,0)^{+-}_{\frac{rs}{2}-1}}$ and $\lambda_{(\emptyset,\emptyset,\emptyset,\emptyset,0)^{-+}_{\frac{rs}{2}-1}}$ in the notation*

$$\Psi \quad = \quad \Delta(\lambda_{(\emptyset,\emptyset,\emptyset,\emptyset,0)^{+-}_{\frac{rs}{2}-1}}, \lambda_{(\emptyset,\emptyset,\emptyset,\emptyset,0)^{-+}_{\frac{rs}{2}-1}}) \ . \qquad (4.18)$$

This automatically implies the following theorem.

Theorem 4.K *A neutral singular vector Ψ_0 which satisfies $\Psi_0 = \Delta(\frac{1}{2}, \frac{1}{2})$ at grade 0 in $\widetilde{\mathcal{V}}_{h,q,c}$ is identical to the highest weight vector: $\Psi_0 \equiv |h, q, c\rangle$.*

Proof: Using the standard parametrisation (2.2) we can find $s \in \mathbb{C}$ such that $h_{0,s} = h$, $\Psi_0 \in \widetilde{\mathcal{V}}_{h_{0,s},q,c}$ and hence $\Psi_0 \in \Pi^{\circ}_{0,s}$. Since $\frac{1}{2}\{G^+_{-\frac{1}{2}}, G^-_{-\frac{1}{2}}\}L^{-1}_{-1} = 1$ we find for $\Delta(\frac{1}{2}, \frac{1}{2})$ at grade

0 that $\Delta(\frac{1}{2}, \frac{1}{2}) = |h, q, c\rangle$. Application of theorem 4.I completes the proof. □

The vectors in $\Pi^{\pm}_k$ are uniquely defined up to scalar multiples. The notation $\Psi^{\pm}_k$ shall indicate the normalisation $\lambda_{(\emptyset,\emptyset,\emptyset,\emptyset,1)^{-+}_{k-\frac{3}{2}}} = 1$ for Ψ^+_k and $\lambda_{(\emptyset,\emptyset,\emptyset,\emptyset,1)^{+-}_{k-\frac{3}{2}}} = 1$ for Ψ^-_k.

We have not yet shown that $\Pi^{\circ}_{r,s}$ is indeed always two dimensional. However we will in the following section explicitly construct a basis for this two dimensional space. Beyond the purpose of this paper but of independent interest would be the question whether we have found herewith all singular vectors in the modules $\widetilde{\mathcal{V}}_{h,q,c}$. We can in fact say that due to the two parameters r and s for the uncharged case not only q and t are free parameters but so is also the grade a. We can hence for each h, q and t guarantee to find a solution space at each grade a which is non-trivial but at most two dimensional. Therefore we may denote $\Pi^{\circ}_{r,s}$ by giving the grade $a = \frac{rs}{2}$ only[p]: Π°_a. Similar thoughts solve in the Virasoro case the related problem completely as discussed by Fuchs[9], although the charged singular vectors in the $\mathtt{sc}(2)$ case only depend on one parameter which complicates the problem. Furthermore, we have not yet looked at higher charged singular vectors which may appear in the generalised module. At the end of the following section we will be able to give a solution to both problems.

4.4. Products of singular vector operators

In this subsection we define products of singular vector operators. We start off giving their definition:

Definition 4.L *A singular vector $\Psi_{a,p} \in \widetilde{\mathcal{V}}_{h,q,c}$ at grade a with charge p defines uniquely an operator $\Theta_{a,p} \in \widetilde{\mathtt{sc}(2)}_-$ such that $\Psi_{a,p} = \Theta_{a,p}\,|h, q, c\rangle$. We call this operator a singular vector operator with weight vector $\omega = (h, q, c)^T$ and grade vector $\xi = (a, p, 0)^T$.*

Theorem 4.M *Let us take two singular vector operators Θ_1 and Θ_2 with corresponding weight vectors $\omega_i = (h_i, q_i, c)^T$ at grade $\xi_i = (a_i, p_i, 0)^T$, $(i = 1, 2)$. If*

$$\begin{pmatrix} h_1 \\ q_1 \\ c \end{pmatrix} + \begin{pmatrix} a_1 \\ p_1 \\ 0 \end{pmatrix} = \begin{pmatrix} h_2 \\ q_2 \\ c \end{pmatrix} , \tag{4.19}$$

then the formal Cauchy product $\Theta_2\Theta_1$ is a singular vector operator with weight $(h_1, q_1, c)^T$ at grade $(a_1 + a_2, p_1 + p_2, 0)^T$. Hence, $\Theta_2\Theta_1\,|h_1, q_1, c\rangle$ is a singular vector.

Proof: We take the cut off vectors $(\Theta_2\Theta_1)^M\,|h_1, q_1, c\rangle$. The definition of the Cauchy product implies:

$$(\Theta_2\Theta_1)^M\,|h_1, q_1, c\rangle = \Theta_2^{M+2}\Theta_1^{M+1}\,|h_1, q_1, c\rangle + \mathcal{O}(a_1 + a_2 - M - 1) \ .$$

[p]The spaces Π certainly depend on h, q and t, however we shall omit this dependence in the notation in the same way as for the singular vectors.

Let us take[q] $X \in \mathtt{sc}(2)_+$. Since $\Theta_1 |h_1, q_1, c\rangle$ is singular, there exists a module homomorphism ϕ from $\widetilde{\mathcal{V}}_{h_2,q_2,c}$ into $\widetilde{\mathcal{V}}_{h_1,q_1,c}$ such that $\phi(|h_2, q_2, c\rangle) = \Theta_1 |h_1, q_1, c\rangle$. Hence we have $\Theta_1^{M+1} |h_1, q_1, c\rangle = \phi(|h_2, q_2, c\rangle) + \mathcal{O}(a_1 - M - 2)$:

$$
\begin{aligned}
X(\Theta_2\Theta_1)^M |h_1, q_1, c\rangle &= X\Theta_2^{M+2}\Theta_1^{M+1} |h_1, q_1, c\rangle + \mathcal{O}(a_1 + a_2 - M) \\
&= \phi(X\Theta_2^{M+2} |h_2, q_2, c\rangle) + \mathcal{O}(a_1 + a_2 - M) \\
&= \phi(\mathcal{O}(a_2 - M - 1)) + \mathcal{O}(a_1 + a_2 - M) \\
&= \mathcal{O}(a_1 + a_2 - M) \ .
\end{aligned}
$$

The statement about the grade of the vector is trivial. □

In the following section, theorem 4.M which is on products of singular vector operators will be fundamental to construct singular vectors as product expressions of known singular vectors. The key question will be to find out when we are allowed to take the product, i.e. when the relation (4.19) is true. For this purpose we want to introduce a relation on the weight space:

Definition 4.N *Let Ω denote the set of complex weights: $\Omega = \mathbb{C} \times \mathbb{C} \times \mathbb{C}$, and let Ξ be the space of grades: $\Xi = \mathbb{C} \times \mathbb{Z} \times \{0\}$. The set Υ is defined to be the set of pairs $(\omega, \xi) \in \Omega \times \Xi$ for which there exists a singular vector operator with weight ω at grade ξ. We say $(\omega_1, \xi_1) \in \Upsilon$ is related to $(\omega_2, \xi_2) \in \Upsilon$ if*

$$
\omega_1 + \xi_1 = \omega_2 \ .
$$

In symbols we write: $(\omega_1, \xi_1) \sim (\omega_2, \xi_2)$.

This relation is neither an equivalence nor an ordering relation. It does not even satisfy any of the standard axioms. Nevertheless, it relates those weights for which we can take products of singular vector operators to obtain another singular vector operator. We find the following multiplicative structure:

Theorem 4.O *Let $\theta_i(a_i, b_i)$ be the singular vector operator of the singular vector $\Delta_i(a_i, b_i)$ with weight ω_i and grade ξ_i, where $i \in \{1, 2\}$. $\Theta_k^\pm$ denotes the charged singular vector operator of $\Psi_k^\pm$ with weight $\omega^\pm = (h_k^\pm, q^\pm, c)$ at grade $\xi^\pm$. If $(\omega_1, \xi_1) \sim (\omega_2, \xi_2)$ then:*

$$
\theta_2(a_2, b_2)\theta_1(a_1, b_1) = 2\theta(a_1 a_2, b_1 b_2) \ .
$$

Similarly, we find:

$$
\begin{aligned}
(\omega_{k^+}^+, \xi_{k^+}^+) \sim (\omega_{k^-}^-, \xi_{k^-}^-) &\Rightarrow \Theta_{k^-}^- \Theta_{k^+}^+ = \theta(0, 1) \ , \\
(\omega_{k^-}^-, \xi_{k^-}^-) \sim (\omega_{k^+}^+, \xi_{k^+}^+) &\Rightarrow \Theta_{k^+}^+ \Theta_{k^-}^- = \theta(1, 0) \ .
\end{aligned}
$$

[q] X acting on $\mathcal{O}(a - N)$ may create an additional L_{-1}, thus $X\mathcal{O}(a - N) = \mathcal{O}(a - N + 1)$.

Finally we can write the vector $\Psi_{r,s}$ for $(r,s)\in\mathbb{N}\times 2\mathbb{N}$ in the new notation. From Eq. (3.9) and after normalising suitably[r] we derive

$$\Psi_{r,2} \quad = \quad \Delta_{r,2}\left(\prod_{n=1}^{r}(\frac{q+1}{t}-\frac{r+1}{2}+n),\prod_{n=1}^{r}(\frac{q-1}{t}+\frac{r+1}{2}-n)\right) \quad . \qquad (4.20)$$

Similarly we identify the more general vectors $\Psi_{r,s}$ using the conjecture (3.11):

$$\Psi_{r,s} \quad = \quad \Delta_{r,s}\left(\prod_{n=1}^{r}(\frac{s-rt}{2t}+\frac{q}{t}-\frac{1}{2}+n),\prod_{n=1}^{r}(-\frac{s-rt}{2t}+\frac{q}{t}+\frac{1}{2}-n)\right) \quad . \qquad (4.21)$$

5. Product expressions for singular vectors

We use the operators $\Theta_k^{\pm}$ which are the singular vector operators of $\Psi_k^{\pm}$. It is trivial to verify the relations:

$$h_{r,s}(t,q) \quad = \quad h^{+}_{-\frac{s-rt}{2t}-\frac{q}{t}}(t,q) \quad , \qquad (5.1)$$

$$h_{r,s}(t,q) \quad = \quad h^{-}_{-\frac{s-rt}{2t}+\frac{q}{t}}(t,q) \quad , \qquad r,s\in\mathbb{C}. \qquad (5.2)$$

This implies that $\Theta^{+}_{-\frac{s-rt}{2t}-\frac{q}{t}}(t,q)\,|h_{r,s}(t,q),q,c(t)\rangle$ and $\Theta^{-}_{-\frac{s-rt}{2t}+\frac{q}{t}}(t,q)\,|h_{r,s}(t,q),q,c(t)\rangle$ are singular vectors in $\tilde{\mathcal{V}}_{h_{r,s}(t,q),q,c(t)}$. Furthermore, we can verify weight relations linking $\Theta_k^{+}(t,q)$ and $\Theta_k^{-}(t,q)$:

$$h_k^{+}(t,q)+k \quad = \quad h^{-}_{k+2\frac{q+1}{t}}(t,q+1) \quad , \qquad (5.3)$$

$$h_k^{-}(t,q)+k \quad = \quad h^{+}_{k-2\frac{q-1}{t}}(t,q-1) \quad . \qquad (5.4)$$

According to the product theorem 4.M we find the neutral singular vectors:

$$\Theta^{-}_{-\frac{s-rt}{2t}+\frac{q}{t}+\frac{2}{t}}(t,q+1)\Theta^{+}_{-\frac{s-rt}{2t}-\frac{q}{t}}(t,q)\,|h_{r,s}(t,q),q,c(t)\rangle \quad , \qquad (5.5)$$

$$\Theta^{+}_{-\frac{s-rt}{2t}-\frac{q}{t}+\frac{2}{t}}(t,q-1)\Theta^{-}_{-\frac{s-rt}{2t}+\frac{q}{t}}(t,q)\,|h_{r,s}(t,q),q,c(t)\rangle \quad . \qquad (5.6)$$

We can now multiply these vectors again alternating with operators of the form Θ^{+} and Θ^{-}, where we have to choose the correct grade for the operators according to[s] Eqs. (5.3) and (5.4):

$$\left(\prod_{m=0}^{u-1}{}^{>}\ \Theta^{-}_{-\frac{s-rt}{2t}+\frac{q}{t}+\frac{2+2m}{t}}(t,q+1)\Theta^{+}_{-\frac{s-rt}{2t}-\frac{q}{t}+\frac{2m}{t}}(t,q)\right)|h_{r,s}(t,q),q,c(t)\rangle \quad , \qquad (5.7)$$

$$\left(\prod_{m=0}^{u-1}{}^{>}\ \Theta^{+}_{-\frac{s-rt}{2t}-\frac{q}{t}+\frac{2+2m}{t}}(t,q-1)\Theta^{-}_{-\frac{s-rt}{2t}+\frac{q}{t}+\frac{2m}{t}}(t,q)\right)|h_{r,s}(t,q),q,c(t)\rangle \quad . \qquad (5.8)$$

[r] Note that we have changed the normalisation of $\Psi_{r,s}$. ¿From now on we always use this normalisation unless stated otherwise.

[s] Since the order in the product is significant, we define $\prod_{m=a}^{b}{}^{>}\ f(m)=f(b)f(b-1)\ldots f(a+1)f(a)$.

These singular vectors are at grade $-u\frac{s-rt}{t}+\frac{2u^2}{t}$. For $u=\frac{s}{2}$ they turn out to be at grade $\frac{rs}{2}$. This allows us to construct for all $t,q\in\mathbb{C}$ two linearly independent vectors in the space $\Pi^\circ_{r,s}$ which proves that $\Pi^\circ_{r,s}$ is always two dimensional:

Theorem 5.A *For the space $\Pi^\circ_{r,s}$ of uncharged singular vectors in $\widetilde{\mathcal{V}}_{h_{r,s}(t,q),q,c(t)}$ at grade $\frac{rs}{2}$ we can give the two basis vectors ($r,s\in\mathbb{C}$)*

$$\Delta_{r,s}(0,1) \;=\; \frac{1}{2^{\frac{s}{2}-1}}\prod_{m=0}^{\frac{s}{2}-1}{}^{>}\;\Theta^-_{-\frac{s-rt}{2t}+\frac{q}{t}+\frac{2+2m}{t}}(t,q+1)\Theta^+_{-\frac{s-rt}{2t}-\frac{q}{t}+\frac{2m}{t}}(t,q)\,|h_{r,s},q,c\rangle\;, \qquad (5.9)$$

$$\Delta_{r,s}(1,0) \;=\; \frac{1}{2^{\frac{s}{2}-1}}\prod_{m=0}^{\frac{s}{2}-1}{}^{>}\;\Theta^+_{-\frac{s-rt}{2t}-\frac{q}{t}+\frac{2+2m}{t}}(t,q-1)\Theta^-_{-\frac{s-rt}{2t}+\frac{q}{t}+\frac{2m}{t}}(t,q)\,|h_{r,s},q,c\rangle\;. \qquad (5.10)$$

We can now very easily identify the singular vector $\Psi_{r,s}\in\mathcal{V}_{h_{r,s}(t,q),q,c(t)}$ where $(r,s)\in\mathbb{N}\times 2\mathbb{N}$. We construct the linear combination of the vectors (5.9) and (5.10) which leads us to the coefficient (4.21):

$$\begin{aligned}\Psi_{r,s} \;&=\; \Delta_{r,s}\left(\prod_{n=1}^{r}\left(\frac{s-rt}{2t}+\frac{q}{t}-\frac{1}{2}+n\right),\prod_{n=1}^{r}\left(-\frac{s-rt}{2t}+\frac{q}{t}+\frac{1}{2}-n\right)\right) \qquad (5.11)\\ &=\; \prod_{n=1}^{r}\left(\frac{s-rt}{2t}+\frac{q}{t}-\frac{1}{2}+n\right)\Delta_{r,s}(1,0)+\prod_{n=1}^{r}\left(-\frac{s-rt}{2t}+\frac{q}{t}+\frac{1}{2}-n\right)\Delta_{r,s}(0,1)\;.\end{aligned}$$

The uncharged vector (5.11) is singular in $\widetilde{\mathcal{V}}_{h_{r,s}(t,q),q,c(t)}$ at grade $\frac{rs}{2}$ and it is the only one, up to scalar multiples, with the required coefficient $\Lambda_2(r,s)$ [Eq. (3.11)]. Hence, for $r\in\mathbb{N}$ and $s\in 2\mathbb{N}$ it has to be the singular vector $\Psi_{r,s}\in\mathcal{V}_{h_{r,s}(t,q),q,c(t)}$, based on our conjecture (3.11).

The methods used to obtain the product expression (5.11) were quite different from the methods used in the Virasoro case[11]. This was mainly due to the coefficients of $\Psi_{r,2}$ not being polynomials in the grade r. We could not tell if the analytic continuation exists and if it does, in terms of what functions it does exist. The expression (5.11) can now give us an answer to this problem. The singular vector $\Psi_{r,s}$ is a linear combination of two infinite vectors, both having polynomial coefficients. The linear combination coefficients are products which we can continue analytically using Γ-functions. This proves as well that we can continue $\Psi_{r,2}$ in the usual sense, by writing it in terms of the basis $\widetilde{\mathcal{C}}_{h_{r,2}(t,q),q,c(t)}$. The coefficients will be linear functions of the products in (5.11). The analytic continuation of $\prod_{n=1}^{r}(\frac{q+1}{t}-\frac{r+1}{2}+n)$ is given by $\frac{\Gamma(\frac{q+1}{t}+\frac{r+1}{2})}{\Gamma(\frac{q+1}{t}-\frac{r-1}{2})}$ and for $\prod_{n=1}^{r}(\frac{q-1}{t}+\frac{r+1}{2}-n)$ we find $\frac{\Gamma(\frac{q-1}{t}+\frac{r+1}{2})}{\Gamma(\frac{q-1}{t}-\frac{r-1}{2})}$. This enables us to define the analytic continuation of $\Psi_{r,2}$ using the vectors (5.9) and (5.10) and choosing a suitable normalisation:

Definition 5.B *For $r\in\mathbb{C}$ we define*

$$\begin{aligned}\tilde{\Psi}_{r,2} \;&=\; \frac{\Gamma(\frac{q+1}{t}+\frac{r+1}{2})}{\Gamma(\frac{q-1}{t}+\frac{r+1}{2})}\Delta_{r,2}(1,0)+\frac{\Gamma(\frac{q+1}{t}-\frac{r-1}{2})}{\Gamma(\frac{q-1}{t}-\frac{r-1}{2})}\Delta_{r,2}(0,1)\\ &=\; \Delta_{r,2}\Big(\frac{\Gamma(\frac{q+1}{t}+\frac{r+1}{2})}{\Gamma(\frac{q-1}{t}+\frac{r+1}{2})},\frac{\Gamma(\frac{q+1}{t}-\frac{r-1}{2})}{\Gamma(\frac{q-1}{t}-\frac{r-1}{2})}\Big)\;.\end{aligned}$$

$\tilde{\Theta}_{r,2}$ is defined to be the singular vector operator of $\tilde{\Psi}_{r,2}$. For $r \in \mathbb{N}$, $\tilde{\Psi}_{r,2}$ is proportional to $\Psi_{r,s}$.

Similarly to the Virasoro case we will now derive a product expression for $\Psi_{r,s}$ using operators of the form $\Theta_{r,2}$ only. We find $h_{r,s}(t,q) = h_{-\frac{s-rt}{t}+\frac{2}{t},2}(t,q)$ thus $\tilde{\Theta}_{-\frac{s-rt}{t}+\frac{2}{t},2}\,|h_{r,s}(t,q),q,c(t)\rangle$ is a singular vector. We verify the relation $h_{r,2}(t,q)+r = h_{r+\frac{4}{t},2}(t,q)$. It allows us to construct products of the operators $\tilde{\Theta}_{r,2}$. If we apply this relation successively, we obtain a neutral singular vector at grade $\frac{rs}{2}$:

$$\tilde{\Theta}_{\frac{s+rt}{t}-\frac{2}{t},2}\tilde{\Theta}_{\frac{s+rt}{t}-\frac{6}{t},2}\ldots\tilde{\Theta}_{-\frac{s-rt}{t}+\frac{6}{t},2}\tilde{\Theta}_{-\frac{s-rt}{t}+\frac{2}{t},2}\,|h_{r,s}(t,q),q,c(t)\rangle \qquad \in \Pi^{\circ}_{r,s} \tag{5.12}$$

To identify the vector (5.12) for $r \in \mathbb{N}$ and $s \in 2\mathbb{N}$ in $\Pi^{\circ}_{r,s}$ we have to know as usual the first two coefficients. We use the definition (5.B) and the multiplication rules[t] of theorem 4.O:

$$\begin{aligned}&\tilde{\Theta}_{\frac{s+rt}{t}-\frac{2}{t},2}\tilde{\Theta}_{\frac{s+rt}{t}-\frac{6}{t},2}\ldots\tilde{\Theta}_{-\frac{s-rt}{t}+\frac{6}{t},2}\tilde{\Theta}_{-\frac{s-rt}{t}+\frac{2}{t},2}\,|h_{r,s}(t,q),q,c(t)\rangle\\ &= 2^{\frac{s}{2}-1}\theta_{r,s}\left(\frac{\Gamma(\frac{q}{t}+\frac{s+rt}{2t}+\frac{1}{2})}{\Gamma(\frac{q}{t}-\frac{s-rt}{2t}+\frac{1}{2})},\frac{\Gamma(\frac{q}{t}+\frac{s-rt}{2t}+\frac{1}{2})}{\Gamma(\frac{q}{t}-\frac{s+rt}{2t}+\frac{1}{2})}\right)\ .\end{aligned} \tag{5.13}$$

Thus, this identifies the vector (5.12) to be $\Psi_{r,s}$ for $r \in \mathbb{N}$ and $s \in 2\mathbb{N}$. As a matter of fact we just grouped the linear combination of products in the expression (5.11) in a product of linear combinations. This was easily done due to the antisymmetric character of $\Theta^{\pm}_{k}$:

$$\begin{aligned}\Psi_{r,s} \;=\; &\frac{1}{2^{\frac{s}{2}-1}}\frac{\Gamma(\frac{q}{t}-\frac{s-rt}{2t}+\frac{1}{2})}{\Gamma(\frac{q}{t}+\frac{s-rt}{2t}+\frac{1}{2})}\tilde{\Theta}_{\frac{s+rt}{t}-\frac{2}{t},2}\tilde{\Theta}_{\frac{s+rt}{t}-\frac{6}{t},2}\cdots\\ &\ldots\tilde{\Theta}_{-\frac{s-rt}{t}+\frac{6}{t},2}\tilde{\Theta}_{-\frac{s-rt}{t}+\frac{2}{t},2}\,|h_{r,s}(t,q),q,c(t)\rangle\ .\end{aligned} \tag{5.14}$$

By extending the algebra $\mathtt{sc}(2)$ to $\widetilde{\mathtt{sc}(2)}$ we have introduced inverse operators of L_{-1}. Does the extended algebra contain in addition other inverse operators of elements in $\mathtt{sc}(2)$? We will find that $\widetilde{\mathtt{sc}(2)}$ does contain inverse operators for $\tilde{\Theta}_{r,2}$. As the weight relation $h_{r,2}(t,q)+r = h_{-r,2}(t,q)$ suggests, $\tilde{\Theta}_{-r,2}\tilde{\Theta}_{r,2}\,|h_{r,2}(t,q),q,c(t)\rangle$ is singular at grade 0. Again, we need to classify this singular vector in Π°_{0}:

$$\begin{aligned}\tilde{\Theta}_{-r,2}\tilde{\Theta}_{r,2} \;&=\; \theta_{-r,2}\left(\frac{\Gamma(\frac{q+1}{t}-\frac{r-1}{2})}{\Gamma(\frac{q-1}{t}-\frac{r-1}{2})},\frac{\Gamma(\frac{q+1}{t}+\frac{r+1}{2})}{\Gamma(\frac{q-1}{t}+\frac{r+1}{2})}\right)\theta_{r,2}\left(\frac{\Gamma(\frac{q+1}{t}+\frac{r+1}{2})}{\Gamma(\frac{q-1}{t}+\frac{r+1}{2})},\frac{\Gamma(\frac{q+1}{t}-\frac{r-1}{2})}{\Gamma(\frac{q-1}{t}-\frac{r-1}{2})}\right)\\ &=\; 4\frac{\Gamma(\frac{q+1}{t}+\frac{r+1}{2})}{\Gamma(\frac{q-1}{t}+\frac{r+1}{2})}\frac{\Gamma(\frac{q+1}{t}-\frac{r-1}{2})}{\Gamma(\frac{q-1}{t}-\frac{r-1}{2})}\theta_0(\frac{1}{2},\frac{1}{2})\ .\end{aligned}$$

Provided $\tilde{\Theta}_{r,2}$ does not vanish identically[u], we can apply theorem 4.K which leads us to:

$$\tilde{\Theta}^{-1}_{r,2} \;=\; \frac{1}{4}\frac{\Gamma(\frac{q-1}{t}+\frac{r+1}{2})}{\Gamma(\frac{q+1}{t}+\frac{r+1}{2})}\frac{\Gamma(\frac{q-1}{t}-\frac{r-1}{2})}{\Gamma(\frac{q+1}{t}-\frac{r-1}{2})}\tilde{\Theta}_{-r,2}\ . \tag{5.15}$$

[t] Let us recall that $\theta_{r,s}(a,b)$ is the singular vector operator of $\Delta_{r,s}(a,b)$

[u] The roots of $\tilde{\Theta}_{r,2}$ will be considered in a later section.

For the uncharged operators $\Theta_k^{\pm}$ we will be less successful. We find relations $h_k^+(t,q)+k = h_{-k}^-(t,q+1)$ and $h_k^-(t,q)+k = h_{-k}^+(t,q-1)$ which imply that the two operators $\Theta_{-k}^-(t,q+1)\Theta_k^+(t,q)$ and $\Theta_{-k}^+(t,q-1)\Theta_k^-(t,q)$ are singular vector operators at grade 0. However, identifying them in Π_0° gives:

$$\begin{aligned}\Theta_{-k}^-(t,q+1)\Theta_k^+(t,q) &= \theta_0(0,1) \ ,\\ \Theta_{-k}^+(t,q-1)\Theta_k^-(t,q) &= \theta_0(1,0) \ .\end{aligned}$$

These equations cannot be inverted. Nevertheless, we can construct a combination of them which is according to theorem 4.K equal to the identity:

$$\frac{1}{2}\Theta_{-k}^-(t,q+1)\Theta_k^+(t,q)+\frac{1}{2}\Theta_{-k}^+(t,q-1)\Theta_k^-(t,q) = 1 \ .$$

In fact having no inverse operators for the charged singular vectors is not surprising at all. The reason for this is that we extended the algebra by uncharged operators only. In order to include inverse operators for the charged singular vectors we would have had to extend the algebra further, introducing charged extended operators.

The result of this section enables us to name all singular vectors in the generalised module $\widetilde{\mathcal{V}}_{h,q,t}$. As mentioned earlier for the uncharged vectors we can use the two parameters of $h_{r,s}$ to fix h and the grade $a=\frac{rs}{2}$ independently for suitably chosen $r,s\in\mathbb{C}$. Hence the space $\Pi_{r,s}^M$ defines a two dimensional space of uncharged singular vectors in $\widetilde{\mathcal{V}}_{h,q,t}$ at grade a. Let us then assume that $\Psi\in\widetilde{\mathcal{V}}_{h,q,t}$ is singular at grade k with charge different from 0 or ±1. The proof of theorem 4.C can be applied in exactly the same way except that in this case the smallest terms $(\emptyset,\emptyset,\emptyset,\emptyset,r)^{+-}$ and $(\emptyset,\emptyset,\emptyset,\emptyset,r)^{-+}$ both cannot exist due to the charge of the vector. Hence there are no singular vectors in $\widetilde{\mathcal{V}}_{h,q,t}$ with charge different from 0 or ±1. Finally in the module $\widetilde{\mathcal{V}}_{h,q,t}$ we find a $+1$ charged singular vector $\Psi_{k^+}^+ = \Theta_{k^+}^+\,|h,q,c\rangle$ at grade $k^+ = -\frac{q}{t}+\frac{1}{2t}\sqrt{t^2+8th+4q^2}$ and a -1 charged singular vector $\Psi_{k^-}^- = \Theta_{k^-}^-\,|h,q,c\rangle$ at grade $k^- = \frac{q}{t}+\frac{1}{2t}\sqrt{t^2+8th+4q^2}$ since in both cases $k^\pm$ was chosen such that $h=h_{k^\pm}^\pm$. For given grade $\hat{k}$ we look at the module $\widetilde{\mathcal{V}}_{h+k^+,q+1,t}$ and take its uncharged singular vector operator $\theta_{\hat{k}-k^+}(1,0)$ at grade $\hat{k}-k^+$. Then the vector $\theta_{\hat{k}-k^+}(1,0)\Theta_{k^+}^+\,|h,q,c\rangle$ is singular in $\widetilde{\mathcal{V}}_{h,q,t}$ with charge $+1$ at the given grade $\hat{k}$. It is important to note that we could not have taken the operator $\theta_{\hat{k}-k^+}(0,1)$ since $\theta_{\hat{k}-k^+}(0,1)\Theta_{k^+}^+\,|h,q,c\rangle=0$. Similarly we can construct a -1 charged singular vector at grade $\hat{k}$. This gives us all singular vectors in the Verma module $\widetilde{\mathcal{V}}_{h,q,t}$.

Theorem 5.C *In the Verma module $\widetilde{\mathcal{V}}_{h,q,t}$ we find at each grade $a\in\mathbb{C}$ exactly a two dimensional space Π_a° of uncharged singular vectors, a one dimensional space Π_a^+ of $+1$ charged singular vectors and a one dimensional space Π_a^- of -1 charged singular vectors. This is a complete list of all singular vectors in $\widetilde{\mathcal{V}}_{h,q,t}$.*

6. Relations among the uncharged singular vector operators

So far, we considered one class of BSA analogue operators only. These were the operators $\Theta_{r,2}$ for which we can give the rather simple expressions (2.4) for $r \in \mathbb{N}$. We extended these operators analytically in the previous section. Using the fusion procedure described in our earlier paper [6] we can also find expressions for the second class of BSA singular vectors $\Psi_{1,s}$. But since we have to start the procedure using a grade 2 singular vector, the vectors $\Psi_{1,s}$ turn out to be more complicated. The explicit formulae for $\Psi_{1,s}$ are given in App. A for the fusion parameter $\eta = 0$. We now want to repeat briefly the results to continue $\Theta_{1,s}$ analytically. Analogously to Eqs. (3.9) and (3.10) we find if λ_0 is normalised to 1:

$$\Lambda_2(1,s) \;=\; \frac{1}{2}\Big[\frac{q+\frac{s}{2}}{q-\frac{s}{2}} - 1\Big] \;=\; \frac{\frac{s}{2}}{q-\frac{s}{2}} \ , \tag{6.1}$$

$$\Lambda_3(1,s) \;=\; \frac{q+1}{t}\frac{s}{2} \ . \tag{6.2}$$

This identifies $\Psi_{1,s}$ uniquely in $\Pi^\circ_{1,s}$ and we obtain after rescaling:

$$\Psi_{1,s} \;=\; \Delta_{1,s}\Big(\frac{q+\frac{s}{2}}{t}, \frac{q-\frac{s}{2}}{t}\Big) \ . \tag{6.3}$$

Thus, the analytic continuation turns out to be rather simple. We define for $s \in \mathbb{C}$:

$$\tilde{\Psi}_{1,s} \;=\; \frac{q+\frac{s}{2}}{t}\Delta_{1,s}(1,0) + \frac{q-\frac{s}{2}}{t}\Delta_{1,s}(0,1) \ , \tag{6.4}$$

and we let $\tilde{\Theta}_{1,s}$ denote the singular vector operator of $\tilde{\Psi}_{1,s}$. We proceed exactly as for $\tilde{\Psi}_{r,2}$. The key points are the weight relations

$$h_{r,s}(t,q) \;=\; h_{1,(s-rt)+t}(t,q) \ , \tag{6.5}$$

$$h_{1,s}(t,q) + \frac{s}{2} \;=\; h_{1,s+2t}(t,q) \ . \tag{6.6}$$

Hence, the neutral vector

$$\tilde{\Theta}_{1,s+t(r-1)}\tilde{\Theta}_{1,s+t(r-1)-2t}\dots\tilde{\Theta}_{1,s-t(r-1)+2t}\tilde{\Theta}_{1,s-t(r-1)}\,|h_{r,s}(t,q),q,c(t)\rangle \tag{6.7}$$

is singular at grade $\frac{rs}{2}$. Again, by comparing the two first coefficients with (4.21), for $r \in \mathbb{N}$ and $s \in 2\mathbb{N}$, we can identify (6.7) to be $\Psi_{r,s}$:

$$\Psi_{r,s} \;=\; \frac{1}{2^{r-1}}\tilde{\Theta}_{1,s+t(r-1)}\tilde{\Theta}_{1,s+t(r-1)-2t}\dots\tilde{\Theta}_{1,s-t(r-1)+2t}\tilde{\Theta}_{1,s-t(r-1)}\,|h_{r,s},q,c\rangle \ . \tag{6.8}$$

Finally, we construct the inverse operator of $\tilde{\Theta}_{1,s}$ based on the weight relation: $h_{1,s}(t,q)+\frac{s}{2} = h_{1,-s}(t,q)$. We obtain:

$$\begin{aligned}
\tilde{\Theta}_{1,-s}\tilde{\Theta}_{1,s} \;&=\; 2\theta_{1,-s}\Big(\frac{q-\frac{s}{2}}{t}, \frac{q+\frac{s}{2}}{t}\Big)\theta_{1,s}\Big(\frac{q+\frac{s}{2}}{t}, \frac{q-\frac{s}{2}}{t}\Big) \\
&=\; 4\frac{q^2-\frac{s^2}{4}}{t^2}\theta_0\Big(\frac{1}{2}, \frac{1}{2}\Big) \\
\Rightarrow \qquad \tilde{\Theta}^{-1}_{1,s} \;&=\; \frac{1}{4}\frac{t^2}{q^2-\frac{s^2}{4}}\tilde{\Theta}_{1,-s} \ .
\end{aligned} \tag{6.9}$$

In the following we investigate relations among the two types of BSA analogue operators $\tilde{\Theta}_{r,2}$ and $\tilde{\Theta}_{1,s}$. The key observations are the following identities among the conformal weights, which can be checked easily:

$$\begin{aligned} h_{r,2}(t,q) + r &= h_{1,rt+t+2}(t,q) \ , && (6.10)\\ h_{r,2}(t,q) + r &= h_{1,-rt+t-2}(t,q) \ , && (6.11)\\ h_{1,s}(t,q) + \frac{s}{2} &= h_{\frac{s+2}{t}+1,2}(t,q) \ , && (6.12)\\ h_{1,s}(t,q) + \frac{s}{2} &= h_{-\frac{s-2}{t}-1,2}(t,q) \ . && (6.13) \end{aligned}$$

In the usual manner, EQS. (6.10)-(6.13) lead us to an identity among the operators $\tilde{\Theta}_{r,2}$ and $\tilde{\Theta}_{1,s}$. For $r, s \in \mathbb{C}$ we have:

$$\tilde{\Theta}_{1,rt+t+2}(t,q)\ \tilde{\Theta}_{r,2}(t,q) = \tilde{\Theta}_{r+2,2}(t,q)\ \tilde{\Theta}_{1,rt+t-2}(t,q) \ . \qquad (6.14)$$

This relation is equivalent to:

$$\begin{aligned} \tilde{\Theta}^{-1}_{1,rt-t+2}(t,q)\ \tilde{\Theta}_{r,2}(t,q) &= \tilde{\Theta}_{r-2,2}(t,q)\ \tilde{\Theta}^{-1}_{1,rt-t-2}(t,q) \ , && (6.15)\\ \tilde{\Theta}_{1,s+4}(t,q)\ \tilde{\Theta}_{\frac{s-t+2}{t},2}(t,q) &= \tilde{\Theta}_{\frac{s+t+2}{t},2}(t,q)\ \tilde{\Theta}_{1,s}(t,q) \ , && (6.16)\\ \tilde{\Theta}^{-1}_{1,-s+4}(t,q)\ \tilde{\Theta}_{\frac{-s+t+2}{t},2}(t,q) &= \tilde{\Theta}_{\frac{-s-t+2}{t},2}(t,q)\ \tilde{\Theta}^{-1}_{1,-s}(t,q) \ . && (6.17) \end{aligned}$$

We can summarise these equations using a diagram[v] which visualises the commuting products of the operators $\tilde{\Theta}_{r,2}$ and $\tilde{\Theta}_{1,s}$:

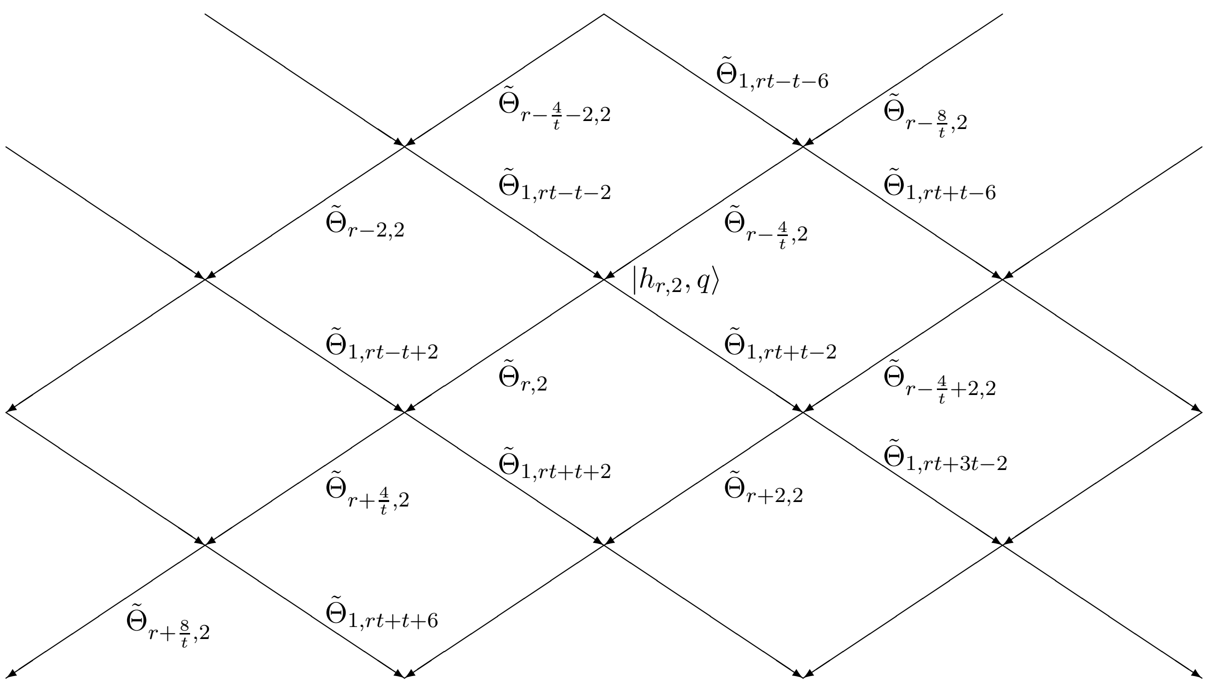

FIG. i Commuting products of operators of the form $\tilde{\Theta}_{r,2}$ and $\tilde{\Theta}_{1,s}$.

[v] Note that this diagram contains operators of the type $\tilde{\Theta}_{r,2}$ and $\tilde{\Theta}_{1,s}$ only but does not consider the remaining vectors in $\Pi^{\circ}_{r,2}$ and $\Pi^{\circ}_{1,s}$.

As we will see in the next section, the operators $\tilde{\Theta}_{r,2}$ and $\tilde{\Theta}_{1,s}$ may vanish for certain points (h, q, c). Besides FIG. i suggests for the uncharged singular vectors a structure similar to the Virasoro case[11].

Exactly as for $\tilde{\Theta}_{r,2}$ and $\tilde{\Theta}_{1,s}$ we can find inverse operators for the analytic continuation of $\Theta_{r,s}$. We define the analytic continuation[w] of (5.11):

$$\bar{\Psi}_{r,s} \quad = \quad \frac{\Gamma\left(\frac{s+rt}{2t} + \frac{q}{t} + \frac{1}{2}\right)}{\Gamma\left(\frac{s-rt}{2t} + \frac{q}{t} + \frac{1}{2}\right)} \Delta_{r,s}(1,0) + \frac{\Gamma\left(-\frac{s-rt}{2t} + \frac{q}{t} + \frac{1}{2}\right)}{\Gamma\left(-\frac{s+rt}{2t} + \frac{q}{t} + \frac{1}{2}\right)} \Delta_{r,s}(1,0) \ . \tag{6.18}$$

As usual, we denote the singular vector operator of $\bar{\Psi}_{r,s}$ by $\bar{\Theta}_{r,s}$. In order to find the inverse operator of $\bar{\Theta}_{r,s}$ we need to multiply it by an operator $\bar{\Theta}_{r',s'}$ such that $h_{r,s}(t,q) + \frac{rs}{2} = h_{r',s'}(t,q)$ and in addition $\frac{rs}{2} + \frac{r's'}{2} = 0$. These weight conditions have four solutions for (r', s'): $(-r, s)$, $(r, -s)$, $(\frac{s}{t}, -rt)$ and $(-\frac{s}{t}, rt)$. These solutions do not define four different inverse operators, in fact the operators corresponding to these solutions are mutually proportional. Again, the inverse operator exists only at the points where $\bar{\Psi}_{r,s}$ does not vanish identically. Identifying the vectors in Π_0° as usual leads to:

$$\bar{\Theta}_{r,s}^{-1} = \frac{1}{4}\bar{\Theta}_{-r,s} \ . \tag{6.19}$$

7. Roots of the product expression for $\Psi_{r,s}$

¿From now on we consider again $r \in \mathbb{N}$ and $s \in 2\mathbb{N}$. We obtained for $\Psi_{r,s}$ the product expression (5.11):

$$\Psi_{r,s} \quad = \quad \underbrace{\prod_{n=1}^{r}(\frac{s-rt}{2t} + \frac{q}{t} - \frac{1}{2} + n)}_{\epsilon^+_{r,s}(t,q)} \Delta_{r,s}(1,0) + \underbrace{\prod_{n=1}^{r}(-\frac{s-rt}{2t} + \frac{q}{t} + \frac{1}{2} - n)}_{\epsilon^-_{r,s}(t,q)} \Delta_{r,s}(0,1) \ .$$

If we follow a curve in the (t, q) plane on which $\epsilon^-_{r,s}(t, q) = 0$ we find that $\Delta_{r,s}(1, 0)$ is singular in $\mathcal{V}_{h_{r,s}(t,q),q,c(t)}$. Similarly for $\epsilon^+_{r,s}(t, q) = 0$, $\Delta_{r,s}(0, 1)$ is a singular vector in $\mathcal{V}_{h_{r,s}(t,q),q,c(t)}$. The linear system which determines the coefficients of a singular vector can be written with polynomial entries, hence, at an intersection point of the curves $\epsilon^+_{r,s}(t, q) = 0$ and $\epsilon^-_{r,s}(t, q) = 0$ we observe that both, $\Delta_{r,s}(1, 0)$ and $\Delta_{r,s}(0, 1)$ are singular vectors in $\mathcal{V}_{h_{r,s}(t,q),q,c(t)}$ at the same grade $\frac{rs}{2}$. At these intersection points the product expression (5.11) vanishes identically and we have $\Delta_{r,s}(1, 0)$ and $\Delta_{r,s}(0, 1)$ spanning its tangent space. In this section we investigate further to find out, where these intersection points are. In the following section we look closer at the tangent space at an intersection point and we give an explicit example.

Definition 7.A *We define:*

$$\epsilon^{\pm}_{r,s}(t,q) \quad = \quad \prod_{n=1}^{r}(\pm\frac{s-rt}{2t} + \frac{q}{t} \mp \frac{1}{2} \pm n) \ . \tag{7.1}$$

[w]Note that by definition $\bar{\Psi}_{r,2}$ and $\tilde{\Psi}_{r,2}$ are proportional only and not identical.

If we assume $\epsilon^+_{r,s}(t,q)=0$, this implies that there exists a $k\in\mathbb{N}_{\frac{1}{2}}, \frac{1}{2}\leq k\leq r-\frac{1}{2}$, such that $\frac{s-rt}{2t}+\frac{q}{t}-\frac{1}{2}=-(k+\frac{1}{2})$. A simple calculation shows that we then obtain $h_{r,s}(t,q)=h^+_k(t,q)$. Similar considerations for $\epsilon^-_{r,s}(t,q)$ lead to:

Theorem 7.B

$$\begin{aligned}\epsilon^+_{r,s}(t,q)=0 \quad&\Leftrightarrow\quad \exists k\in\mathbb{N}_{\frac{1}{2}},\ \ \tfrac{1}{2}\leq k\leq r-\tfrac{1}{2}:\ \ h_{r,s}(t,q)=h^+_k(t,q)\ \ ,\\ \epsilon^-_{r,s}(t,q)=0 \quad&\Leftrightarrow\quad \exists k\in\mathbb{N}_{\frac{1}{2}},\ \ \tfrac{1}{2}\leq k\leq r-\tfrac{1}{2}:\ \ h_{r,s}(t,q)=h^-_k(t,q)\ \ .\end{aligned}$$

The curves with vanishing functions $\epsilon^+_{r,s}(t,q)$ or $\epsilon^-_{r,s}(t,q)$ turn out to be the representations, where in addition to the uncharged singular vector in $\mathcal{V}_{h_{r,s}(t,q),q,c(t)}$ we have at least one charged singular vector. To investigate these representations further, we parametrise the conformal weight h by $a,\tilde{t}$ and q. Feigin and Fuchs treated the Virasoro case in exactly the same way[8]:

$$h \;=\; \frac{a^2-q^2-\tilde{t}^2}{4\tilde{t}}\quad,\quad a,\tilde{t},q\in\mathbb{C}\ \ . \tag{7.2}$$

Here $\tilde{t}$ is the rescaled t: $\tilde{t}=\frac{t}{2}$. If we assume $h=h_{r,s}$ we find that the point $(r,\tilde{s})$ is an integer pair solution of the linear equation $\tilde{s}=\tilde{t}r-a$, where $\tilde{s}=\frac{s}{2}$. Also, $h=h^+_k$ has two roots: $k^+=-\frac{q}{2\tilde{t}}+\frac{a}{2\tilde{t}}$ and $k^-=\frac{q}{2\tilde{t}}+\frac{a}{2\tilde{t}}$. We assume that $k^+=-\frac{q}{2\tilde{t}}+\frac{a}{2\tilde{t}}$ is in $\mathbb{N}_{\frac{1}{2}}$. These are exactly the representations we want to look at, if only k^+ is in the range $\frac{1}{2}\leq k^+\leq r-\frac{1}{2}$; then $\epsilon^+_{r,s}(t,q)=0$. Hence, besides the neutral singular vector $\Psi_{r,s}$ at grade $r\tilde{s}$ there is a positive-charged singular vector $\Psi^+_{k^+}$ at grade k^+. Starting from the vector $\Psi^+_{k^+}$ as highest weight vector embedded in $\mathcal{V}_{h_{r,s}(t,q),q,c(t)}$ we find that its weight is parametrised by $a'=a+1$ and $q'=q+1$. Hence we obtain a neutral singular vector $\Psi_{r',s'}$ in this embedded module, by solving the equation for integer r' and $\tilde{s}'$:

$$\tilde{s}'-\tilde{s} \;=\; \tilde{t}(r'-r)-1\ \ . \tag{7.3}$$

Eq. (7.3) has at least one solution: $r'_0=r$ and $\tilde{s}'_0=\tilde{s}-1$. Only for $\tilde{t}\in\mathbb{Q}$ we can find more solutions. In this case with $\tilde{t}=\frac{u}{v}$ and u,v coprime, we find the additional solutions: $r'_n=r'_0+nv$ and $\tilde{s}'_n=\tilde{s}'_0+nu$ where $n\in\mathbb{Z}$. If $r'_n\tilde{s}'_n>0$ then we call the corresponding singular vector $\Psi_{r'_n,s'_n}$. Again, we take $\Psi_{r'_n,s'_n}$ as our new highest weight vector, embedded in the original module. The embedded module has the parameters $a''=a+2\tilde{s}'_n+1$ and $q''=q+1$. We try to find out if this embedded module contains a negative-charged singular vector. For this purpose we construct the combination $k^-_n=\frac{q+1}{2\tilde{t}}+\frac{a+2\tilde{s}'_n+1}{2\tilde{t}}$. For $(r'_0,\tilde{s}'_0)$ we find $k^-_0=-k^++r$ which is even valid for $\tilde{t}\notin\mathbb{Q}$, and if $\tilde{t}\in\mathbb{Q}$ we obtain the additional solutions: $k^-_n=-k^++r+nv$. If $k^+\in\mathbb{N}_{\frac{1}{2}}$ and $\frac{1}{2}\leq k^+\leq r-\frac{1}{2}$ we have $k^-_0\in\mathbb{N}_{\frac{1}{2}}$. If we were dealing with the Virasoro case, we would believe that the product of the three[x] operators $\Theta^-_{k^-_0}$, $\Theta_{r'_0,\tilde{s}'_0}$ and $\Theta^+_{k^+}$ leads to a neutral singular vector in $\mathcal{V}_{h_{r,s}(t,q),q,c(t)}$: $\Theta^-_{k^-_0}\Theta_{r'_0\tilde{s}'_0}\Theta^+_{k^+}\,|h_{r,s}(t,q),q,c(t)\rangle$ which is at grade $k^++k^-_0+r'_0\tilde{s}'_0=r\tilde{s}$. However, as a consequence of theorem (7.B), the operator $\Theta_{r'_0,\tilde{s}'_0}$

[x]For $\tilde{s}'_0=0$ we only take $\Theta^-_{k^-_0}\Theta^+_{k^+}$.

is of the form $\Delta_{r'_0,\tilde{s}'_0}(0,1)$ since $\frac{1}{2} \le k^+ \le r'_0 - \frac{1}{2}$ except for the case that both conditions of theorem (7.B) hold. Therefore in the case $\tilde{s}'_0 \neq 0$ we may find $\Theta_{r'_0\tilde{s}'_0}\Theta^+_{k^+}\,|h_{r,s}(t,q),q,c(t)\rangle \equiv 0$ and otherwise we obtain $\Psi_{r,s}$. In FIG. ii we shall indicate this by dotted lines meaning that the shown connexions may be trivial. And conversely, if we assume we have the singular vector $\Psi_{r,s}$ and Ψ_{k^+} and we want to write $\Psi_{r,s}$ as a product of singular vector operators in $\mathcal{V}_{h_{r,s}(t,q),q,c(t)}$. We want this product from the left to the right to consist of a negative-charged operator, an optional product of uncharged operators and a positive-charged operator $\Theta^+_{k^+}$. For $\tilde{t} \notin \mathbb{Q}$ there is no other possibility than the one described above and hence necessarily $\frac{1}{2} \le k^+ \le r - \frac{1}{2}$. If $\tilde{t} \in \mathbb{Q}$, we can in addition find $\Theta_{r'_n,s'_n}$ or a product $\Theta_{r''_n,s''_n}\Theta_{r'_{m,n},s'_{m,n}}$ where $r''_{m,n} = r'_0 + (m+n)v$ and $\tilde{s}''_{m,n} = -\tilde{s}'_0 + (m-n)u$. An analysis of both cases shows that we find necessarily $\frac{1}{2} \le k^+ \le r - \frac{1}{2}$. However, in the same way as before theorem 7.B implies that the product may vanish. The same arguments can be applied for $\epsilon^-_{r,s}(t,q)$. We can summarise this important result in the following theorem:

Theorem 7.C *The representations with $\epsilon^+_{r,s}(t,q) = 0$ or $\epsilon^-_{r,s}(t,q) = 0$ are exactly the ones which can be summarised in the diagrams:*

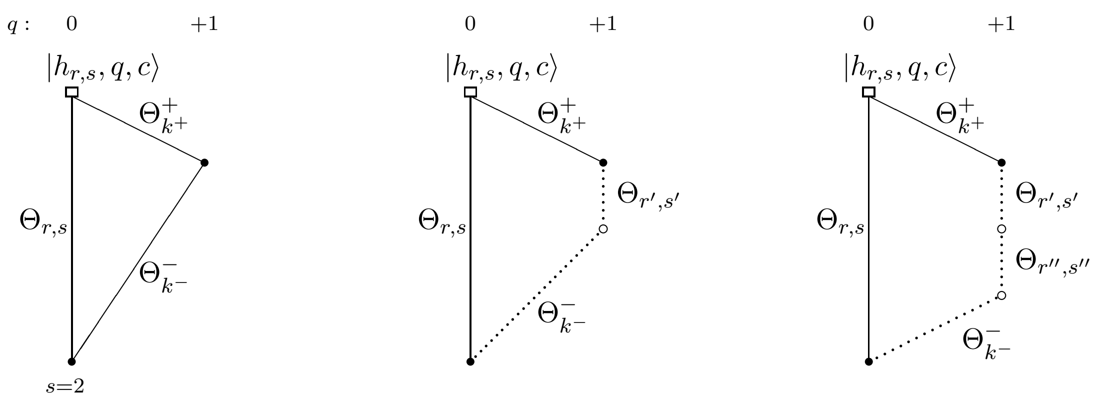

FIG. ii Representations with $\epsilon^+_{r,s}(t,q) = 0$.

Similarly for the case $\epsilon^-_{r,s}(t,q) = 0$.

At the intersection points of the curves $\epsilon^+_{r,s}(t,q) = 0$ and $\epsilon^-_{r,s}(t,q) = 0$ we have representations containing the uncharged singular vector and both one $+1$ charged and one -1 charged singular vector. In the embedding diagram starting at the highest weight vector and following either of the fermionic lines, we may for both fermionic singular vectors reach the grade $\frac{rs}{2}$. However these two vectors are not the same according to the expression (5.11) for $\Psi_{r,s}$. We want to call the representations at the intersection points of $\epsilon^+_{r,s}(t,q) = 0$ and $\epsilon^-_{r,s}(t,q) = 0$ *degenerate representations* and the intersection points themselves shall be called *points of degeneration*. We have herewith classified all representations for which (5.11) identically vanishes and produces two linearly independent singular vectors. The feature of having two linearly independent neutral singular vectors at the same grade is so far unique in the case of conformal algebras considered in the literature. The implications that the degenerate representations are

exactly the ones given in FIG. ii rely on the conjecture of the coefficient $\Lambda_2(r,s)$ [EQ. (3.11)]. This conjecture is based on the proven expressions for $\Psi_{r,2}$ and $\Psi_{1,s}$, on computer evidence for different values of r and s and as well on consistency calculations of the product expressions for $\Psi_{r,s}$ using known singular vectors. Hence we can find plenty of cases for which $\Lambda_2(r,s)$ and the degeneration is proven. Among them we will give one explicit example in the following section. As this example shows, the $N=2$ embedding diagrams conjectured independently by Kiritsis[12], Dobrev[5] and Matsuo[14] are wrong. Moreover, it is an immediate consequence of the results of this paper to find out which products of embedding homomorphisms are trivial. We discuss the embedding diagrams for the $\mathtt{sc}(2)$ algebra in a forthcoming paper[7].

8. Degenerate singular vectors

In the previous section we have found that for some particular cases both vectors in $\Pi^{\circ}_{r,s}$ happen to be finite and we obtain two linearly independent neutral singular vectors in $\mathcal{V}_{h_{r,s}(t,q),q,c(t)}$ at the same grade $\frac{rs}{2}$. On the other hand, we gave in ref. [6] explicit expressions for $\Psi_{r,2}$. Using the tangent space of $\Psi_{r,2}$ and without using the knowledge of the previous section, we can understand the fact that $\Psi_{r,2}$ is a linear combination of two generalised singular vectors which both happen to be finite at a point of degeneration. We may assume that the expression of $\Psi_{r,s}$ is given with polynomial coefficients. If it vanishes identically for a particular pair (t,q), we then divide the singular vector components, which are polynomials in t and q, by the common root. It is easy to see that we obtain at most two linearly independent vectors at such a point.

Let us consider a polynomial vector field $\Psi(t,q)$ over a two dimensional differentiable manifold parametrised by (t,q). Suppose there exists a point (t_0,q_0) at which the vector field vanishes. Following an arbitrary linear path through this point, $\alpha t+\beta q=\alpha t_0+\beta q_0$, we find that we can factorise this root from the vector field in order to obtain the derivative[y] $v_{\alpha,\beta}(t_0,q_0)$: $\Psi_{\alpha,\beta}=(\alpha t+\beta q-\alpha t_0-\beta q_0)v_{\alpha,\beta}$. It corresponds to the partial derivative $\alpha\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial q}$:

$$\begin{aligned}(\alpha\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial q})\Psi_{\alpha,\beta}(t,q)\Big|_{(t_0,q_0)} &= \Big((\alpha^2+\beta^2)+(\alpha t+\beta q-\alpha t_0-\beta q_0)\,(\alpha\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial q})\Big)\,v_{\alpha,\beta}(t,q)\Big|_{(t_0,q_0)}\\ &=(\alpha^2+\beta^2)v_{\alpha,\beta}(t_0,q_0)\ .\end{aligned}$$

Here $(\alpha^2+\beta^2)$ does not vanish. The tangent space is two dimensional and hence $v_{\alpha,\beta}(t_0,q_0)$ lies in a two dimensional vector space.

Consequently in the case where the polynomial expression for $\Psi_{r,s}$ vanishes, it can describe at most two linearly independent singular vectors at the same grade with exactly the same charge. We now give an explicit example for this new feature which does not appear in the Virasoro case or the $N=1$ superconformal case, where the underlying manifold is just one dimensional.

As an example we choose $\Psi_{3,2}$ as given by EQ. (B1) in APP. B using the standard basis. We

[y] In the case where the root $\alpha t_0+\beta q_0$ is contained more than once the same argument can be applied successively.

calculate the singular vectors at the point[z] $(t_0, q_0) = (1, 0)$ where (B1) obviously vanishes.

$$\begin{aligned}
\Delta_{3,2}(0,1) &= \tilde{\Theta}^-_{\frac{3}{2}+\frac{q+1}{t}}(t,q+1)\tilde{\Theta}^+_{\frac{3}{2}-\frac{q+1}{t}}(t,q)\,|h_{3,2}(t,q),q,c(t)\rangle \quad , &(8.1)\\
\Delta_{3,2}(1,0) &= \tilde{\Theta}^+_{\frac{3}{2}+\frac{1-q}{t}}(t,q-1)\tilde{\Theta}^-_{\frac{3}{2}-\frac{1-q}{t}}(t,q)\,|h_{3,2}(t,q),q,c(t)\rangle \quad . &(8.2)
\end{aligned}$$

These expressions make it clear that the vectors $\Delta_{3,2}(1,0)$ and $\Delta_{3,2}(0,1)$ are products of the operators $\Theta^{\pm}_{\frac{1}{2}}$ and $\Theta^{\pm}_{\frac{5}{2}}$ on the lines $q-t+1=0$ and $q+t-1=0$.

$$\begin{aligned}
\Delta_{3,2}(0,1)\,|_{q=t-1} &= \Theta^-_{\frac{5}{2}}(t,t)\Theta^+_{\frac{1}{2}}(t,t-1)\left|\frac{t-1}{2},t-1,3-3t\right\rangle \quad , &(8.3)\\
\Delta_{3,2}(1,0)\,|_{q=1-t} &= \Theta^+_{\frac{5}{2}}(t,-t)\Theta^-_{\frac{1}{2}}(t,1-t)\left|\frac{t-1}{2},1-t,3-3t\right\rangle \quad . &(8.4)
\end{aligned}$$

We use Eqs. (B2) - (B5) given in ref. [6] to determine $\Delta_{3,2}(0,1)\,|_{q=t-1}$ and $\Delta_{3,2}(1,0)\,|_{q=1-t}$. After normalising suitably we find:

$$\begin{aligned}
\Delta_{3,2}(0,1)\,|_{q=t-1} &= \Big\{4L_{-1}^3+12L_{-1}^2T_{-1}-2L_{-1}^2G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}-8tL_{-2}L_{-1}+8L_{-1}T_{-1}^2\\
&\quad -6L_{-1}T_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}-4(2t+5)L_{-1}T_{-2}+2(t-1)L_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\\
&\quad -8tL_{-2}T_{-1}+2(t+1)L_{-2}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}+4t(t-1)L_{-3}-4T_{-1}^2G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\\
&\quad -4(t-4)T_{-2}T_{-1}+4tT_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}+(3t+5)T_{-2}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}+4(t+1)\\
&\quad (t+4)T_{-3}-2(t-1)(t+1)G^+_{-\frac{1}{2}}G^-_{-\frac{5}{2}}\Big\}\left|\frac{t-1}{2},t-1,3-3t\right\rangle \quad , &(8.5)\\
\Delta_{3,2}(1,0)\,|_{q=1-t} &= \Big\{2L_{-1}^2G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}-6L_{-1}T_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}-2(t-1)L_{-1}G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}}\\
&\quad -2(t+1)L_{-2}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}+4T_{-1}^2G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}+4tT_{-1}G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}}+(3t+5)\\
&\quad T_{-2}^2G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}+2(t-1)(t+1)G^+_{-\frac{5}{2}}G^-_{-\frac{1}{2}}\Big\}\left|\frac{t-1}{2},1-t,3-3t\right\rangle \quad . &(8.6)
\end{aligned}$$

If we evaluate (B1) on the two lines $q-t+1=0$ and $q+t-1=0$ we verify the proportionality:

$$\begin{aligned}
\Psi_{3,2}\,|_{q=t-1} &= -(t-2)\Delta_{3,2}(0,1)\,|_{q=t-1} \quad , &(8.7)\\
\Psi_{3,2}\,|_{q=1-t} &= -(t-2)\Delta_{3,2}(1,0)\,|_{q=1-t} \quad . &(8.8)
\end{aligned}$$

Finally, considering the point $(t_0, q_0) = (1, 0)$ at which $\Psi_{3,2}$ vanishes leads us to two linearly independent singular vectors. Following the line $q-t+1=0$ into $(1,0)$ we find as singular vector $\Delta_{3,2}(0,1)\,|_{t=1,q=0}$ which corresponds to the partial derivative of $\Psi_{3,2}(t,q)$ with $\alpha=1$ and $\beta=-1$ in $(1,0)$:

$$\Delta_{3,2}(0,1)\,|_{t=1,q=0} = \Big\{4L_{-1}^3+12L_{-1}^2T_{-1}-2L_{-1}^2G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}-8L_{-2}L_{-1}+8L_{-1}T_{-1}^2$$

[z] Let us remark that this point belongs to the unitary series; in fact it is the trivial one-dimensional representation.

$$
\begin{aligned}
&-6L_{-1}T_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} - 28L_{-1}T_{-2} - 8L_{-2}T_{-1} + 4L_{-2}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} \\
&-4T^2_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} + 12T_{-2}T_{-1} + 4T_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}} + 8T_{-2}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} \\
&+40T_{-3}\Big\} \, |0,0,0\rangle \quad . \qquad (8.9)
\end{aligned}
$$

Similarly, we can follow the line $q + t - 1 = 0$ into $(1,0)$ for which we end up with the singular vector $\Delta_{3,2}(1,0) \mid_{t=1,q=0}$, corresponding to the partial derivative of $\Psi_{3,2}(t,q)$ with $\alpha = 1$ and $\beta = 1$ in $(1,0)$:

$$
\begin{aligned}
\Delta_{3,2}(1,0) \mid_{t=1,q=0} \;=\; &\Big\{2L^2_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} - 6L_{-1}T_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} - 4L_{-2}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} + 4T^2_{-1} \\
&G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} + 4T_{-1}G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}} + 8T^2_{-2}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\Big\} \, |0,0,0\rangle \quad . \qquad (8.10)
\end{aligned}
$$

These two vectors are linearly independent and span the whole tangent space of $\Psi_{3,2}$ at $(t = 1, q = 0)$. Following any other direction does not give any further information but linear combinations of these two singular vectors.

9. Conclusions

We defined analytic extensions of the $N = 2$ (untwisted) Verma modules for which we showed that they contain at each grade two linearly independent uncharged singular vectors and one $+1$ and one -1 charged singular vector. We constructed these singular vectors explicitly using analytic continuations of the BSA analogue vectors and the charged singular vectors known from ref. [6]. This extended structure which is apparently shared at least by superconformal algebras and Kac-Moody algebras has in our opinion not obtained enough attention by the literature and should be studied in more detail. Our conjecture of the coefficient $\Lambda_2(r,s)$ for the singular vectors $\Psi_{r,s}$ allowed us to give product expressions for all singular vectors of the algebra $\mathtt{sc}(2)$. This leads generically to a Virasoro like structure for the uncharged singular vectors. However there are points at which the whole two dimensional uncharged space of singular vectors of the generalised module lies in the original Verma module and leads to two linearly independent uncharged singular vectors at the same grade. For this important implication of the conjecture we gave an explicit example for confirmation. This disproves the existing literature about the $N = 2$ embedding diagrams[12, 5, 14] and shows a feature of superconformal algebras which had not been discovered so far. We made clear where we disagree with the existing literature about $N = 2$ superconformal embedding diagrams which we will clarify in a forthcoming publication[7].

Appendix

A The singular vectors $\Psi_{1,s}$

We can also write the vectors $\Psi_{1,s}$ for $s \in 2\mathbb{N}$ ($s \neq 2$) as a sum over partitions using the fusion method with $\eta = 0$:

$$
\begin{aligned}
\Psi_{1,s} &= (2t,(t-2)q,0,0) \sum_{\substack{j=2 \\ j \text{ even}}}^{s} \sum_{\substack{n_1+\ldots+n_j=\frac{s}{2} \\ n_i \in \mathbb{N}_{\frac{1}{2}}}} E'_{n_j+\frac{1}{2}}(\frac{s}{2}) T_{n_{j-1}+\frac{1}{2}}(\frac{s}{2}-n_j) \\
&\quad E_{n_{j-2}+\frac{1}{2}}(n_1+\ldots+n_{j-2}) \ldots\ldots T_{n_3+\frac{1}{2}}(n_1+n_2+n_3) E_{n_2+\frac{1}{2}}(n_1+n_2) \\
&\quad T_{n_1+\frac{1}{2}}(n_1) \begin{pmatrix} 2t \\ (t-2)q \\ 0 \\ 0 \end{pmatrix} |h_{1,s}(t,q),q,c(t)\rangle \quad .
\end{aligned}
$$

The four-by-four matrices $E_k(n)$, $T_k(r)$ and $E'_k(n)$ are given by:

$$
\begin{aligned}
E'_k(n) &= \begin{pmatrix} e^k_{11}(n) & e^k_{12}(n) & e^k_{13}(n) & e^k_{14}(n) \\ e^k_{21}(n) & e^k_{22}(n) & e^k_{23}(n) & e^k_{24}(n) \\ 0 & 0 & \gamma(n)\delta_{k,1} & 0 \\ 0 & 0 & 0 & \gamma(n)\delta_{k,1} \end{pmatrix} \quad , \quad k \in \{1,2\} \quad , \\
E'_k(n) &= 0 \quad , \quad k \geq 3 \quad , \\
E_k(n) &= \frac{1}{\gamma(n)} E'_k(n) \quad , \quad k \in \mathbb{N} \quad , \\
T_k(r) &= \begin{pmatrix} \delta_{k,1} & 0 & 0 & 0 \\ 0 & \delta_{k,1} & 0 & 0 \\ t^k_{31}(r) & t^k_{32}(r) & t^k_{33}(r) & t^k_{34}(r) \\ t^k_{41}(r) & t^k_{42}(r) & t^k_{43}(r) & t^k_{44}(r) \end{pmatrix} \quad , \quad k \in \{1,2\} \quad , \\
T_k(r) &= 0 \quad , \quad k \geq 3 \quad , \\
\gamma(n) &= 4[t(n+1-\frac{s}{2})+2(\frac{s}{2}-1)][t(n-1)+2(\frac{s}{2}-1)][n-\frac{s}{2}] \quad , \\
\gamma^{\pm}(r) &= (2q \pm 2rt \mp st \pm s \pm t)(2q \pm 2rt \pm s \mp t) \quad ,
\end{aligned}
$$

$$
\begin{aligned}
e^1_{11}(n) &= \frac{-1}{nt}\{[\{4[(t-2)s+2t+4]q-(t^3+4t^2+12t-16)s-(t^2-8t+4)s^2 \\
&\quad -8(t+2)q^2-2t^3-4t^2-8t-16\}nt+2[3(t-2)s-4q-t^2+4t+12]n^2t^2 \\
&\quad +4(st-s-t+2)(s-t-2)q+2(st-2s+t^2+4)(t+2)q^2+(t^2+4)(t+2)st \\
&\quad -(t-2)s^3t-8n^3t^3-8q^3t-8s^3t]L_{-1}+q[(t^2+6t+4)(t-2)s-(t-2)s^3t \\
&\quad +4t^2+16t+16-[(t^2+4)(t-2)s+2t^3+12t^2+8t+16]n+2(t^2+4)n^2t \\
&\quad +2(t^2-4)q^2]]T_{-1}-4q[[(t-2)s+2t+4]nt+(st-s-t+2)(s-t-2) \\
&\quad -2n^2t^2-2q^2t]G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\} \quad , \\
e^1_{12}(n) &= \frac{4}{n}\{[(t-2)s+t^2+4t+4-4nt-4q]qL_{-1}-[(t-2)sn+2nt+4n-(q^2+s)
\end{aligned}
$$

$$
\begin{aligned}
& (t+2) - 2n^2t + s^2]T_{-1} + 4q^2 G^+_{-\frac{1}{2}} G^-_{-\frac{1}{2}}\} \ , \\
e^1_{13}(n) \ = \ & \frac{-2}{nt}[(3t^2-3t-2)s - (t-1)s^2 + 2t^2 + 4t)q + 2[(t-2)s + 3t + 2]q^2 \\
& -(4q + st - 2s + 2t + 4)(2q + t - 1)nt + 2(2q + t - 1)n^2t^2 + (t+2)(t-1)st \\
& -(t-1)s^2t + 4q^3]G^+_{-\frac{1}{2}} \ , \\
e^1_{14}(n) \ = \ & \frac{2}{nt}[(3t^2-3t-2)s - (t-1)s^2 + 2t^2 + 4t)q - 2[(t-2)s + 3t + 2]q^2 \\
& +(4q - st + 2s - 2t - 4)(2q - t + 1)nt + 2(2q - t + 1)n^2t^2 - (t+2)(t-1)st \\
& +(t-1)s^2t + 4q^3]G^-_{-\frac{1}{2}} \ ,
\end{aligned}
$$

$$
\begin{aligned}
e^1_{21}(n) \ = \ & \frac{1}{2nt^2}\Big[4\{2q[3s(t-2) - t^2 + 12] - 4q^2 - s^2 + 4s + t^2 - 4\}n^2t^2 + [(s-t-2)st \\
& -2(t+2)q^2](st - 2s + t^2 + 4)(t-2)q + 2(4q^3t + 8q^3 + 4^2s - 8q^2t - 8q^2 \\
& -4qs^2t + 4qs^2 + 2qst^2 + 4qst - 16qs + 2qt^3 + 4qt^2 + 8qt + 16q + s^3 - 6s^2 \\
& -st^2 + 12s + 2t^2 - 8)(t-2)nt - 4(st - s - t + 2)(s - t - 2)(t-2)q^2 \\
& +8(t-2)q^4t - 32n^3qt^3\Big]L_{-1} + \frac{1}{2nt^2}\Big[2[(3t^2+16)(t-2)s + (3t-4)(t-2)s^2 \\
& +2(t-2)^2q^2 - 2t^3 + 8t^2 + 8t + 32]n^2t - 8[2(t-2)s + t^2 + 8]n^3t^2 - (2q^2s \\
& (t-2)^2 - 4q^2t^2(t+3) - 16q^2 + s^3t^2 - 2s^3t + s^2t^3 - 4s^2t(t-1) + 12st^2 + 8st \\
& -4t^3 - 16t(t+1)](t-2)n[s^2t(t-2) - st^2(t+4) + 8s(t+1) - 4(t+2)^2] \\
& (t-2)q^2 - 2(t+2)(t-2)^2q^4 + 16n^4t^3\Big]T_{-1} - \frac{1}{nt^2}\Big[(4q^2s - 8q^2t - 8q^2 + s^3 \\
& -6s^2 - st^2 + 12s + 2t^2 - 8)(t-2)nt - 2(4q^2 + s^2 - 4s - t^2 + 4)n^2t^2 \\
& -2(st - s - t + 2)(s - t - 2)(t-2)q^2 + 4(t-2)q^4t\Big]G^+_{-\frac{1}{2}}G^+_{-\frac{1}{2}} \ , \\
e^1_{22}(n) \ = \ & \frac{[2n^2t^2 + nt(2-s)(t-2) - 2q^2(t-2)][4nt + 4q - st + 2s - (t-2)^2]}{nt}L_{-1} \\
& +q\frac{[2(t+2-s)\frac{s}{n} + st^2 + 4s - 2t^2 - 8(t+1)](t-2)n - 2(t^2+4)n^2t + 2(t^2-4)q^2}{nt} \\
& T_{-1} + 4q\frac{(s-2)(t-2)nt + 2(t-2)q^2 - 2n^2t^2}{nt}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} \ , \\
e^1_{23}(n) \ = \ & \frac{2}{nt^2}\Big[2\{2[3(t-2)s - t^2 + 2t + 8]q + (3t^2 - 13t + 2)s + (t+1)s^2 - 4(t-3)q^2 - 2t^2 \\
& +16t - 8\}n^2t^2 - 2(t+2-s)(t-1)(t-2)qst + [16q^3 + 4q^2s(t-3) + 16q^2(t+1) \\
& -2qs^2(t-2) + 2qst^2 - 8qs(t-1) + 4qt(t+4) - s(t+1) - s^2t(t-9) + 4s(s-1) \\
& -18st + 4t(t+2)](t-2)nt - 8(2q + t - 1)n^3t^3 + 2[s^2(t-1) - 3st(t-1) + 2s \\
& -2t(t+2)](t-2)q^2 - 4(st - 2s + 3t + 2)(t-2)q^3 - 8(t-2)q^4\Big]G^+_{-\frac{1}{2}} \ , \\
e^1_{24}(n) \ = \ & \frac{2}{nt^2}\Big[2\{2[3(t-2)s - t^2 + 2t + 8]q - (3t^2 - 13t + 2)s - (t+1)s^2 + 4(t-3)q^2 + 2t^2 \\
& -16t + 8\}n^2t^2 - 2(t+2-s)(t-1)(t-2)qst + [16q^3 - 4q^2s(t-3) - 16q^2(t+1)
\end{aligned}
$$

$$-2qs^2(t-2) + 2qst^2 - 8qs(t-1) + 4qt(t+4) + s(t+1) + s^2t(t-9) - 4s(s-1)$$
$$+18st - 4t(t+2)](t-2)nt - 8(2q-t+1)n^3t^3 - 2[s^2(t-1) - 3st(t-1) + 2s$$
$$-2t(t+2)](t-2)q^2 - 4(st-2s+3t+2)(t-2)q^3 + 8(t-2)q^4\Big]G^-_{-\frac{1}{2}} \ ,$$

$$\begin{aligned}
e^2_{11}(n) &= \frac{2n^2t^2 - nst^2 + 2nst - 2nt^2 - 4nt + 2q^2t + 4q^2 - s^2t + st^2 + 2st}{nt^2} \\
&\quad [4t(t-1)(G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}} - G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}}) + 8t(G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}T_{-1} - L_{-1}T_{-1}) - 4t^2L^2_{-1} \\
&\quad +2t^3L_{-2} + (t^2-4)T^2_{-1} + 4t(t+1)T_{-2}] \ , \\
e^2_{12}(n) &= \frac{2q}{nt}[4t(t-1)(G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}} - G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}}) - 8tT_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} - (t^2-4)T^2_{-1} \\
&\quad -4t(t+1)T_{-2} + 4t^2L^2_{-1} + 8tL_{-1}T_{-1} - 2t^3L_{-2}] \ , \\
e^2_{13}(n) &= \frac{2}{nt}[2n^2t^2 - 4nqt - nst^2 + 2nst - 2nt^2 - 4nt + 2q^2t + 4q^2 + qst - 2qs + qt^2 \\
&\quad +4qt + 4q - s^2t + st^2 + 2st] \\
&\quad [(t-1)G^+_{-\frac{3}{2}} + 2T_{-1}G^+_{-\frac{1}{2}}] \ , \\
e^2_{14}(n) &= \frac{2}{nt}[2n^2t^2 + 4nqt - nst^2 + 2nst - 2nt^2 - 4nt + 2q^2t + 4q^2 - qst + 2qs - qt^2 \\
&\quad -4qt - 4q - s^2t + st^2 + 2st] \\
&\quad [(t-1)G^-_{-\frac{3}{2}} - 2T_{-1}G^-_{-\frac{1}{2}}] \ ,
\end{aligned}$$

$$\begin{aligned}
e^2_{21}(n) &= -q\frac{4n(2n-s)t^2 + 8nst + 4nt(t^2-4) - 2q^2(t^2-4) + s^2t(t-2) - st(t^2-4)}{4nt^3} \\
&\quad \{4t[(t-1)(G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}} - G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}) + 2T_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} - tL^2_{-1} - 2L_{-1}T_{-1}] + 2t^3L_{-2} \\
&\quad +(t^2-4)T^2_{-1} + 4t(t+1)T_{-2}\} \ , \\
e^2_{22}(n) &= \frac{n(2n-s)t^2 + 2nst + 2nt(t-2) - 2q^2(t-2)}{2nt^2} \\
&\quad \{4t[(t-1)(G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}} - G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}) \\
&\quad +2T_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} - tL^2_{-1} - 2L_{-1}T_{-1}] + 2t^3L_{-2} + (t^2-4)T^2_{-1} + 4t(t+1)T_{-2}\} \ , \\
e^2_{23}(n) &= \frac{2}{nt^2}[8n^3t^3 - 16n^2qt^2 - 6n^2st^2(t-2) - 2n^2t^2(t^2+12) - 8nq^2t^2 + 16nq^2t + 8nqst^2 \\
&\quad -16nqst - 8nqt^3 + 32nqt + ns^2t^3 - 4ns^2t^2 + 4ns^2t + nst^4 + 4nst^2 - 16nst \\
&\quad -2nt^3(t+2) + 8nt(t+2) + 4q^3(t^2-4) + 2q^2st(t-4) + 8q^2s + 2q^2t^2(t+2) \\
&\quad -8q^2(t+2) - 2qs^2t(t-2) + 2qst(t^2-4)][(t-1)G^+_{-\frac{3}{2}} + 2T_{-1}G^+_{-\frac{1}{2}}] \ , \\
e^2_{24}(n) &= \frac{-2}{nt^2}[8n^3t^3 + 16n^2qt^2 - 6n^2st^2(t-2) - 2n^2t^2(t^2+12) - 8nq^2t^2 + 16nq^2t - 8nqst^2 \\
&\quad +16nqst + 8nqt^3 - 32nqt + ns^2t^3 - 4ns^2t^2 + 4ns^2t + nst^4 + 4nst^2 - 16nst \\
&\quad -2nt^3(t+2) + 8nt(t+2) - 4q^3(t^2-4) + 2q^2st(t-4) + 8q^2s + 2q^2t^2(t+2) \\
&\quad -8q^2(t+2) + 2qs^2t(t-2) - 2qst(t^2-4)][(t-1)G^-_{-\frac{3}{2}} - 2T_{-1}G^-_{-\frac{1}{2}}] \ ,
\end{aligned}$$

$$
\begin{aligned}
t^1_{31}(r) &= -\frac{2[(t-2)s-4rt-t^2+4]q+(4q^2-s^2)(t-1)-(t^2-5t+2)s+4(t+1)rt-6t}{\gamma^-(r)}G^-_{-\frac{1}{2}},\\
t^1_{32}(r) &= 4t\frac{2q-t-1}{\gamma^-(r)}G^-_{-\frac{1}{2}}\ ,\\
t^1_{33}(r) &= -2t\frac{(t-2)s-4q-4rt+t^2+2t+8}{\gamma^-(r)}L_{-1}-8t\frac{q-1}{\gamma^-(r)}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\\
&\quad -2\frac{qt^2-4q+4rt-st+2s-2t(t+1)}{\gamma^-(r)}T_{-1}\ ,\\
t^1_{41}(r) &= \frac{2[(t-2)s-4rt-t^2+4]q-(4q^2-s^2)(t-1)+(t^2-5t+2)s-4(t+1)rt+6t}{\gamma^+(r)}G^+_{-\frac{1}{2}},\\
t^1_{42}(r) &= 4t\frac{2q+t+1}{\gamma^+(r)}G^+_{-\frac{1}{2}}\ ,\\
t^1_{44}(r) &= -2t\frac{(t-2)s-4q-4rt+t^2+2t}{\gamma^+(r)}L_{-1}-8t\frac{q+1}{\gamma^+(r)}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\\
&\quad -2\frac{qt^2-4q-4rt+st-2s+2t(t+1)}{\gamma^+(r)}T_{-1}\ ,
\end{aligned}
$$

$$
\begin{aligned}
t^2_{31}(r) &= \frac{4rt-st+2s-t^2-2t-4}{\gamma^-(r)}[(t-1)G^-_{-\frac{3}{2}}-2T_{-1}G^-_{-\frac{1}{2}}]\ ,\\
t^2_{32}(r) &= \frac{4t}{\gamma^-(r)}[(t-1)G^-_{-\frac{3}{2}}-2T_{-1}G^-_{-\frac{1}{2}}]\ ,\\
t^2_{33}(r) &= \frac{-1}{\gamma^-(r)}\{4t[(t-1)(G^+-\frac{1}{2}G^-_{-\frac{3}{2}}-G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}})-2T_{-1}G^+_{\frac{1}{2}}G^-_{-\frac{1}{2}}]-(t^2-4)T^2_{-1}\\
&\quad -4(t+1)tT_{-2}+4t^2L^2_{-1}+8tL_{-1}T_{-1}-2t^3L_{-2}\}\ ,\\
t^2_{41}(r) &= \frac{4rt-st+2s-t^2-2t-4}{\gamma^+(r)}[(t-1)G^+_{-\frac{3}{2}}+2T_{-1}G^+_{-\frac{1}{2}}]\ ,\\
t^2_{42}(r) &= \frac{-4t}{\gamma^+(r)}[(t-1)G^+_{-\frac{3}{2}}+2T_{-1}G^+_{-\frac{1}{2}}]\ ,\\
t^2_{44}(r) &= \frac{-1}{\gamma^+(r)}\{4t[(t-1)(G^+-\frac{1}{2}G^-_{-\frac{3}{2}}-G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}})-2T_{-1}G^+_{\frac{1}{2}}G^-_{-\frac{1}{2}}]-(t^2-4)T^2_{-1}\\
&\quad -4(t+1)tT_{-2}+4t^2L^2_{-1}+8tL_{-1}T_{-1}-2t^3L_{-2}\}\ .
\end{aligned}
$$

B The singular vector $\Psi_{3,2}$

$\Psi_{3,2}$ given in the standard basis can be obtained from ref. [6]:

$$
\begin{aligned}
&\Psi_{3,2}=2^6(q+t)(q-t)\Big(\\
&\quad (q+t-1)\Big\{(q-t-1)(q-1)\Big[t^3L^3_{-1}+t^4(t-1)L_{-3}+3t^2(q+1)L^2_{-1}T_{-1}-2t^3(q+1)\\
&\qquad L_{-2}T_{-1}+t(3q^2+6q-t^2+3)L_{-1}T^2_{-1}-t^2(2qt+3q+4t+3)L_{-1}T_{-2}-2t^4L_{-2}L_{-1}
\end{aligned}
$$

$$
\begin{aligned}
&\quad +t^2(qt+2q+3t+2)(t+1)T_{-3} - t(2q^2t+3q^2+6qt+6q-t^3-t^2+4t+3)T_{-2}T_{-1}\Big] \\
&\quad +\Big[t^2(3q^2-2qt^2-3qt+2q+t^3+t^2-t-1)L_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}} + t(3q^3-2q^2t^2-3q^2t \\
&\quad +3q^2+qt^3+qt^2-2qt-3q-t^4+5t^2+t-3)T_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}} + t(q-t^2-2t+1) \\
&\quad (q-t^2+1)(q-1)G^+_{-\frac{1}{2}}G^-_{-\frac{5}{2}}\Big]\Big\} \\
&+(q-t+1)\Big\{-t^2(3q^2+2qt^2+3qt-2q+t^3+t^2-t-1)L_{-1}G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}} - t(3q^3+2q^2t^2 \\
&\quad +3q^2t-3q^2+qt^3+qt^2-2qt-3q+t^4-5t^2-t+3)T_{-1}G^+_{-\frac{3}{2}}G^-_{-\frac{1}{2}} + t(q+t^2+2t-1) \\
&\quad (q+t^2-1)(q+1)G^+_{-\frac{5}{2}}G^-_{-\frac{1}{2}}\Big\} \qquad \text{(B1)} \\
&+(q-t+1)(q+t-1)\Big\{-t(2q^2-t^4-t^3+2t^2-2)G^+_{-\frac{3}{2}}G^-_{-\frac{3}{2}} \\
&\quad +(q+t+1)(q-t-1)(q+1)(q-1)T^3_{-1}\Big\} \\
&+6t^2q(q+1)(q-1)L_{-1}T_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} + t^3(3q^2-t^2+1)L^2_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} - 2t^2(t-1) \\
&(t+1)(q+1)(q-1)L_{-2}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} + t(3q^4-6q^2+t^4-4t^2+3)T^2_{-1}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} \\
&-t^2q(2q^2t+3q^2+t^3-6t-3)T_{-2}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}}\Big)\Big|-\frac{3}{2}+t-\frac{q-1}{2t}, q, c(t)\Big\rangle \quad .
\end{aligned}
$$

In the same way we can give the operators into which $\Psi_{3,2}(1,0)$ factorises as shown in Sec. 8:

$$\Theta^+_{\frac{1}{2}}(t,q) = tq(q+1)G^+_{-\frac{1}{2}} \quad , \qquad \text{(B2)}$$

$$\Theta^-_{\frac{1}{2}}(t,q) = tq(q-1)G^-_{-\frac{1}{2}} \quad , \qquad \text{(B3)}$$

$$
\begin{aligned}
\Theta^+_{\frac{5}{2}}(t,q) &= 2^6t^5q^5(2q+t+2)(2q+t)(q+t+1)(q+t)(q+1)^5 \\
&\Big(2(2q^2+7qt+6t^2-1)G^+_{-\frac{5}{2}} - G^+_{-\frac{3}{2}}G^+_{-\frac{1}{2}}G^-_{-\frac{1}{2}} \\
&-2(2q+3t-1)L_{-1}G^+_{-\frac{3}{2}} + 4(2q+3t)T_{-1}G^+_{-\frac{3}{2}} \\
&+2L^2_{-1}G^+_{-\frac{1}{2}} - 6L_{-1}T_{-1}G^+_{-\frac{1}{2}} + 4T^2_{-1}G^+_{-\frac{1}{2}} \\
&-2(q+2t+1)L_{-2}G^+_{-\frac{1}{2}} + (3q+6t+5)T_{-2}G^+_{-\frac{1}{2}}\Big) \quad , \qquad \text{(B4)}
\end{aligned}
$$

$$
\begin{aligned}
\Theta^-_{\frac{5}{2}}(t,q) &= 2^6t^5q^5(2q-t-2)(2q-t)(q-t-1)(q-t)(q-1)^5 \\
&\Big(2(2q-3t)(q-2t)G^-_{-\frac{5}{2}} - G^+_{-\frac{1}{2}}G^-_{-\frac{3}{2}}G^-_{-\frac{1}{2}} \\
&+2(2q-3t)L_{-1}G^-_{-\frac{3}{2}} + 4(2q-3t)T_{-1}G^-_{-\frac{3}{2}} \\
&+2L^2_{-1}G^-_{-\frac{1}{2}} + 6L_{-1}T_{-1}G^-_{-\frac{1}{2}} + 4T^2_{-1}G^-_{-\frac{1}{2}} \\
&+2(q-2t)L_{-2}G^-_{-\frac{1}{2}} + (3q-6t-4)T_{-2}G^-_{-\frac{1}{2}}\Big) \quad . \qquad \text{(B5)}
\end{aligned}
$$

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